PHYSICS OF PLASMAS 23, 032706 (2016)

On the quantum Landau collision operator and electron collisions in dense plasmas

Jérôme Daligault[a)]
Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA

(Received 5 February 2016; accepted 4 March 2016; published online 22 March 2016)

The quantum Landau collision operator, which extends the widely used Landau/Fokker-Planck collision operator to include quantum statistical effects, is discussed. The quantum extension can serve as a reference model for including electron collisions in non-equilibrium dense plasmas, in which the quantum nature of electrons cannot be neglected. In this paper, the properties of the Landau collision operator that have been useful in traditional plasma kinetic theory and plasma transport theory are extended to the quantum case. We outline basic properties in connection with the conservation laws, the H-theorem, and the global and local equilibrium distributions. We discuss the Fokker-Planck form of the operator in terms of three potentials that extend the usual two Rosenbluth potentials. We establish practical closed-form expressions for these potentials under local thermal equilibrium conditions in terms of Fermi-Dirac and Bose-Einstein integrals. We study the properties of linearized quantum Landau operator, and extend two popular approximations used in plasma physics to include collisions in kinetic simulations. We apply the quantum Landau operator to the classic test-particle problem to illustrate the physical effects embodied in the quantum extension. We present useful closed-form expressions for the electron-ion momentum and energy transfer rates. Throughout the paper, similarities and differences between the quantum and classical Landau collision operators are emphasized. © *2016 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4944392]

I. INTRODUCTION

The work presented in this paper is part of an effort aimed at developing practical approximations to enable kinetic simulations of dense plasmas under non-equilibrium conditions. This is motivated by recent experiments on warm dense matter and on charged-particle transport in plasmas formed along the compression pathway to ignition in inertial confinement fusion experiments. Indeed, by their nature, warm dense matter experiments produce transient, non-equilibrium conditions, and measurements of equilibrium properties may be misleading if recorded while the plasma species are still out of equilibrium.[1] On the other hand, it is likely that current and future X-ray diagnostics offer the possibility to probe the return to equilibrium of the non-equilibrium states thus created, and provide new information on the nature of interactions in warm dense matter.[2–4] Other recent experiments aimed at measuring the stopping power of charged projectiles in inertial fusion targets[5] and warm dense matter,[6,7] as well as alternative particle-beam inertial fusion designs,[8] can also benefit from non-equilibrium kinetic simulations.

Unlike traditional plasmas, dense plasmas are dense enough and cold enough that the wave-like and fermionic nature of electrons can no longer be neglected. A major challenge to performing non-equilibrium simulations of dense plasmas is to include the quantum nature of conduction electrons in their collisions among themselves and with ions. The state of the art computational methods for modeling dense plasmas is finite-temperature density-functional-theory-based molecular dynamics and quantum Monte-Carlo,[9] which, by construction, represent well the electron-electron and electron-ion correlations in thermal equilibrium. However, electrons are not dynamical in these approaches. As a consequence of the fluctuation-dissipation theorem, it is possible to extract linear transport coefficients like the electrical conductivities from these simulations. However, transient dynamics, time-dependent disturbances, and non-equilibrium dynamics beyond the linear regime are not accessible using these methods. The extension of these microscopic methods, e.g., time-dependent density functional theory, to such dynamical conditions is still in its infancy.[9,10] Until now, the majority of non-equilibrium calculations have been done using classical molecular dynamics, in which quantum effects are included through modifications of the pair potentials used in the classical Newton's equations of motion.[11,12] Another approach, which is the prevalent approach in traditional plasma physics, consists of describing electrons with a kinetic equation that describes the evolution of the electron distribution function in phase-space. While quantum kinetic theory is a mature field,[13,14] detailed quantum kinetic equations remain hard to solve both analytically and numerically. This is true not only of the Kadanoff-Baym equations for the non-equilibrium Green's functions but also of less detailed descriptions like the quantum Boltzmann equation first introduced by Uehling and Uhlenbeck to extend the celebrated Boltzmann equation to the quantum realm.[13,15]

In fact, similar remarks can be made about the inclusion of collisions in classical plasma physics. While fairly detailed kinetic theories exist, e.g., the Lenard-Balescu kinetic equation, the simpler kinetic equation derived by Landau is generally preferred in applications.[16,17] The Landau equation or,

[a)]Electronic mail: daligaul@lanl.gov

 © 2016 AIP Publishing LLC

Reuse of AIP Publishing content is subject to the terms at: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 192.12.184.6 On: Tue, 22 Mar 2016 15:05:36

equivalently, the Fokker-Plank operator is, indeed, the starting point or the underlying model of collisions of a large majority of studies in many areas of plasma physics. There, the underlying plasmas are typically hot and dilute enough that the average particle kinetic energy greatly exceeds the potential energy of interaction. In this weakly coupled regime, collisions, i.e., the interactions of charged particles with the electric and magnetic field fluctuations, cause only small deflections of the velocity vector of plasma particles. The effect of these deflections on the one-particle distribution functions is well described by the Landau operator, which is essentially a diffusion operator in velocity space. Curiously, to our knowledge, the quantum extension of the Landau equation has not been considered as a suitable model of electron collisions in dense plasmas. Under dense plasma conditions such as created in high-energy-density experiments, ions are weakly coupled at high enough temperature but become strongly coupled for temperatures below which their mean kinetic energy is lower than their mean potential energy of interaction; the description of ion collisions with the Landau colllision operator is invalid under such strongly coupled conditions. On the contrary, electrons remain weakly coupled among themselves at all temperatures as a result of their fermionic character (higher kinetic energy states are being populated as the temperature decreases). It is therefore legitimate to explore the possibility to model electron-electron interactions with a Landau-like collision operator that accounts for the quantum nature of electrons. Like the classical Landau operator, the quantum Landau collision operator can be obtained by retaining in the Boltzmann-Uehling-Uhlenbeck collision integral, only the small angle scattering events. To our knowledge, it was first introduced in the literature in 1980 by Danielewicz in the context of heavy-ion collision physics, but, apart from a few appearances in the mathematically oriented literature,[18,19] it has not been utilized in physics. Like its classical counterpart, this model of collisions is interesting since it can be derived from controlled, physically motivated approximations; it incorporates important physics, including the effect of quantum degeneracy on the statistics of collisions; and it is more easily amenable to numerical simulations than other more detailed approximations. For these reasons, the quantum Landau collision operator is a relevant, nontrivial model of electron collisions in non-equilibrium dense palsmas, which can serve as a *reference* to more advanced descriptions, in a way similar to the Thomas-Fermi model with respect to advanced density functional theory descriptions for equation-of-state calculations.

Our primary objective is to extend to the quantum case the properties of the Landau collision operator that have been useful in traditional plasma kinetic theory and plasma transport theory (see, e.g., Ref. 17). The extension is often technically not straightforward, and we therefore give in the appendixes the details of the mathematical derivations and tricks used to this purpose. The resulting closed-form expressions highlighted in the main text, however, are easy to use in either analytical or numerical applications. In addition, throughout the paper, we emphasize the similarities and differences between the quantum and classical Landau collision operator. More precisely, the paper is organized as follows. In Sec. II, the quantum Landau collision operator is introduced and its properties are studied. For completeness, we first recall important properties in connection with the conservation laws, the H-theorem, and the global and local equilibrium distributions. We then express the quantum Landau operator in the form of a non-linear Fokker-Planck operator. This requires introducing three Rosenbluth-like potentials, instead of two Rosenbluth potentials for the classical operator. Practical, closed-form expressions of the potentials are then given in the limit of local thermal equilibrium distribution functions. We then illustrate the physical implications of the quantum corrections on the classic test-particle problem in an equilibrium electron-ion plasma; practical expressions are given for the friction and diffusion coefficients and for the energy loss rate of the test-particle. Finally, we present useful expressions for the electron-ion momentum and energy transfer rates in plasmas consisting of quantum electrons and classical ions. In Sec. III, we extend the previous study to the linearized quantum Landau operator, linearized around local thermal equilibrium. This is motivated by the fact that linearized collision operators are central both in the mathematical treatments of kinetic theories like in the Chapman-Enskog method,[20] and to some advanced numerical algorithms like the δf-method.[21] In this regard, we extend to the quantum case two popular approximations of the linearized collision operator that are used in traditional plasma kinetic simulations.

For convenience, the term quantum Landau-Fokker-Planck collision operator is used throughout the paper and abbreviated with the acronym qLFP to refer to the quantum Landau collision operator or to its Fokker-Planck form.

II. QUANTUM LANDAU COLLISION OPERATOR

To the best of our knowledge, the qLFP collision operator was first discussed by Danielewicz in Ref. 22 in the context of heavy-ion collision physics. The operator was derived for general mutual interactions from the grazing collision approximation of the Boltzmann-Uehling-Uhlenbeck kinetic equation.[23] In the appendix of Ref. 22, the general collision operator was specialized to Coulomb interactions. For completeness, in Appendix A, we give a slightly different derivation starting from the Boltzmann-Uehling-Uhlenbeck collision operator with the dynamically screened Coulomb scattering cross section in the Born approximation. By construction, the qLFP collision operator inherits the assumptions at the basis of the Boltzmann-Uehling-Uhlenbeck operator (e.g., regarding quantum exchange, and diffraction), and we refer the reader to the extensive literature on this equation for more details (in particular, we recommend Ref. 13).

A. Definition

We consider a plasma consisting of N species of non-relativistic charged particles (including ions and electrons) of mass m_a, charge $q_a = Z_a e$ (e is minus the electron charge). Each species a is described by a single-particle phase-space distribution function $f_a(\mathbf{r}, \mathbf{p}, t)$, normalized so that $n_a(\mathbf{r}, t) = \int d\mathbf{p} f_a(\mathbf{r}, \mathbf{p}, t)$ is the number density. For simplicity of exposition of the properties of the qLFP collision operator, which

Reuse of AIP Publishing content is subject to the terms at: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 192.12.184.6 On: Tue, 22 Mar 2016 15:05:36

is the focus of this paper, we assume that collisions among all species are described by a quantum Landau collision operator. In applications to dense plasmas, the qLFP kinetic equation could be restricted to conduction electrons, while another description could be chosen to describe the ion dynamics, in particular, under conditions when ions are strongly coupled. Several schemes can be envisioned in that respect with different levels of sophistication. For instance, a simple model would describe both charged species with qLFP operators assuming classical ions and quantum electrons, and would include the effect of strong Coulomb couplings within the Coulomb logarithms, as is supported by the recently developed effective potential theory of transport.[24–26] A more sophisticated approach would combine a qLFP treatment of the electrons with classical molecular dynamics for the ions; the foundations of such a "kinetic theory molecular dynamics" approach were recently discussed by Graziani *et al.*[27]

Within the Landau approximation, the distribution functions f_a satisfy the kinetic equations

$$\frac{Df_a}{Dt} = \sum_b \mathcal{C}_{ab}(f_a, f_b). \tag{1}$$

Here

$$\frac{Df_a}{Dt} = \frac{\partial f_a}{\partial t} + \frac{\mathbf{p}_a}{m_a} \cdot \frac{\partial f_a}{\partial \mathbf{r}} + \mathbf{F}_a \cdot \frac{\partial f_a}{\partial \mathbf{p}_a} \tag{2}$$

is the streaming operator describing the trajectories in phase-space of species a particles under the influence of the force $\mathbf{F}_a$ (e.g., the plasma mean electric field or an external disturbance). $\mathcal{C}_{ab}(f_a, f_b)$ denotes the qLFP operator of interest in this paper, which describes the effect on f_a of collisions between particles of species a with particles of species b (like-species scattering is described by the term with $b = a$). By dropping dependencies on $(\mathbf{r}, t)$, $\mathcal{C}_{ab}$ is given by

$$\begin{aligned} \mathcal{C}_{ab}[f_a, f_b](\mathbf{p}_a) = \gamma^{ab} \frac{\partial}{\partial \mathbf{p}_a} \cdot \int d\mathbf{p}_b \overleftrightarrow{V}_{ab}(\mathbf{p}_a, \mathbf{p}_b) \\ \times \left\{ \frac{\partial f_a(\mathbf{p}_a)}{\partial \mathbf{p}_a} f_b(\mathbf{p}_b) \left[1 + \delta_b \theta_b f_b(\mathbf{p}_b)\right] \right. \\ \left. - \frac{\partial f_b(\mathbf{p}_b)}{\partial \mathbf{p}_b} f_a(\mathbf{p}_a) \left[1 + \delta_a \theta_a f_a(\mathbf{p}_a)\right] \right\}. \end{aligned} \tag{3}$$

Here,

$$\gamma^{ab} = 4\pi q_a^2 q_b^2 \mu_{ab} \ln\Lambda_{ab},$$

where $\ln\Lambda_{ab}$ is the Coulomb logarithm (see below), and

$$\overleftrightarrow{V}_{ab}(\mathbf{p}_a, \mathbf{p}_b) = \frac{1}{2\mu_{ab} \mathrm{v}_{ab}} \left[\overleftrightarrow{\mathrm{I}} - \frac{\mathbf{v}_{ab}\mathbf{v}_{ab}}{\mathrm{v}_{ab}^2} \right],$$

where $\overleftrightarrow{\mathrm{I}}$ is the identity tensor, $\mu_{ab} = m_a m_b/(m_a + m_b)$ is the reduced mass, and

$$\mathbf{v}_{ab} = \mathbf{v}_a - \mathbf{v}_b, \quad \mathbf{v}_a = \frac{\mathbf{p}_a}{m_a}.$$

In Eq. (3), $\delta_a = -1, 0, 1$ for Fermi-Dirac, Boltzmann, and Bose-Einstein statistics, respectively. The expression (3) includes the classical Landau equation as a special case by setting $\delta_a = \delta_b = 0$. In the majority of applications in plasma physics, the ions can be treated as classical particles $\delta_a = 0$ and the electrons are fermions $\delta_a = -1$. However, for sake of generality, the results presented below are derived irrespective of the particles' statistics. Finally, $\theta_a = \frac{(2\pi\hbar)^3}{g_a}$, where g_a is the spin multiplicity factor of species a ($g_a = 2$ for electrons), so that $d\mathbf{r}d\mathbf{p}/\theta_a$ is the number of available states in the phase volume $d\mathbf{r}d\mathbf{p}$.

B. Discussion

1. Quantum degeneracy effect

The bracket terms $[1 + \delta_a \theta_a f_a]$ and $[1 + \delta_b \theta_b f_b]$ in $\mathcal{C}_{ab}$ account for the quantum statistics. For fermions ($\delta_a = -1$), the Pauli principle requires that no more than $\frac{d\mathbf{r}d\mathbf{p}}{\theta_a}$ particles of species a in the volume $d\mathbf{r}$ can possess momenta in the range $d\mathbf{p}$. The probability of a collision that would result in a particle of species a entering this range is thus reduced in the ratio $[1 - \theta_a f_a]$.[28] For bosons, on the contrary, the presence of a like particle in the range $d\mathbf{p}$ increases the probability that a particle will enter that range in the ratio $[1 + \theta_a f_a]$.

2. Coulomb logarithms

The Coulomb logarithm refers to the integral over momentum transfers $\hbar k$

$$\ln\Lambda_{ab} = \int_0^\infty \frac{dk}{k} \tag{4}$$

that arises in the process of retaining only the small-angle scattering events in the Boltzmann collision operator with the Coulomb scattering law, or, as in Appendix A,

$$\ln\Lambda_{ab} = \int_0^\infty \frac{dk}{k} \left| \frac{1}{\epsilon(k, 0)} \right|^2, \tag{5}$$

when using the screened Coulomb scattering cross section in the Born approximation (here ϵ is the total dielectric function in the random phase approximation[29]). The integral (4) is divergent at both ends of the integration range: at large momentum transfer $\hbar k$, because of the grazing collisions approximation, and at small k because of the infinite range of the bare Coulomb potential (the divergence is regularized by the dielectric function in Eq. (5)). In practice, physically motivated cutoff parameters k_{min} and k_{max} are introduced to regularize the otherwise divergent integral, leading $\ln\Lambda = \ln\left(\frac{k_{max}}{k_{min}}\right)$. We refer to Refs. 30 and 31 for detailed discussions on the choice of cutoffs for dense plasmas, and to Appendix A of Ref. 32 for additional choices. For completeness, we recall here the most popular prescription for typical dense electron-ion plasmas. The logarithm is expressed in the form[30]

Reuse of AIP Publishing content is subject to the terms at: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 192.12.184.6 On: Tue, 22 Mar 2016 15:05:36

$$\ln\Lambda = \frac{1}{2}\ln\left(1+\frac{k_{max}^2}{k_{min}^2}\right),$$

with upper and lower cutoffs given as follows. The minimum k_{min} is set by Coulomb screening.[33] For Λ_{ei}

$$k_{min} = \min(k_{sc}, 1/a),$$

where $a_i = (3/4\pi n_i)^{1/3}$ is the interionic distance and the k_{sc} is the inverse screening length

$$k_{sc}^2 = k_{D,i}^2 + k_e^2,$$

where $k_{D,i}$ is the ionic Debye length

$$k_{D,i}^2 = \frac{4\pi n_i q_i^2}{k_B T_i},$$

and k_e is the Thomas-Fermi screening length

$$k_e^2 = k_{D,e}^2 \frac{\mathcal{Q}_{-\frac{1}{2}}(\beta_e\mu_e)}{\mathcal{Q}_{\frac{1}{2}}(\beta_e\mu_e)} \approx \frac{k_{D,e}^2}{\left(1+T_F^2/T_e^2\right)^{\frac{1}{2}}},$$

in terms of the Fermi-Dirac integral defined below. For Λ_{ee}, $k_{min} = k_e$. The upper limit k_{max} is, under typical dense plasma conditions, set by the characteristic inverse electron deBroglie wavelength of electrons, which is conveniently approximated across degeneracy regimes by

$$k_{max}^2 = \frac{24\pi}{\lambda_{th}^2}\left(1+\frac{T_F^2}{T_e^2}\right)^{\frac{1}{2}},$$

with $\lambda_{th} = \sqrt{2\pi\hbar^2/m_e k_B T_e}$ the thermal deBroglie wavelength.

3. Non-linearity

The quantum operator has a higher-order nonlinearity than its classical counterpart, since the dependence on the distribution functions is cubic in the quantum case and quadratic in the classical case. This leads to extra difficulties to deal with in both the analytical and numerical treatments.

C. Properties

Like its classical counterpart,[17] the qLFP collision operator satisfies physically important properties in connection with the conservation laws and with the concept of irreversibility. These properties can be readily derived assuming that the distribution functions vanish sufficiently fast as $|\mathbf{p}| \to \infty$ to eliminate surface integrals. Although these properties have already been discussed in Ref. 22, we recall them here without proof for the sake of completeness.

1. Local conservation laws

At each space-time point $(\mathbf{r}, t)$, the total number of particles of any species $n_a = \int d\mathbf{p} f_a$, the total momentum $\mathbf{P} = \sum_a \int d\mathbf{p}\, \mathbf{p} f_a$, and the total (kinetic) energy $E = \sum_a \int d\mathbf{p} \frac{\mathbf{p}^2}{2m_a} f_a$ are conserved by collisions. More precisely,

$$\int d\mathbf{p}\, \mathcal{C}_{ab}[f_a, f_b](\mathbf{p}) = 0,$$

$$\int d\mathbf{p}\, \mathbf{p}\, \mathcal{C}_{ab}[f_a, f_b](\mathbf{p}) = -\int d\mathbf{p}\, \mathbf{p}\, \mathcal{C}_{ba}[f_b, f_a](\mathbf{p}),$$

$$\int d\mathbf{p}\, \frac{\mathbf{p}^2}{2m_a}\, \mathcal{C}_{ab}[f_a, f_b](\mathbf{p}) = -\int d\mathbf{p}\, \frac{\mathbf{p}^2}{2m_b}\, \mathcal{C}_{ba}[f_b, f_a](\mathbf{p}),$$

i.e., the local density is not affected by collisions, the momentum transfer rate from species b to species a is equal in magnitude and opposite in direction to that from a to b, and the energy is conserved in a binary collisions between species a and b.

2. H-theorem

Consider the total entropy density s and flux $\mathbf{j_s}$ defined as

$$s(\mathbf{r}, t) = -k_B \sum_a \int \frac{d\mathbf{p}}{\theta_a}\left[\bar{f}_a \ln \bar{f}_a - \delta_a\left(1+\delta_a \bar{f}_a\right)\ln\left(1+\delta_a \bar{f}_a\right)\right]$$

$$\mathbf{j}_s(\mathbf{r}, t) = -k_B \sum_a \int \frac{d\mathbf{p}}{\theta_a}\frac{\mathbf{p}}{m_a}\left[\bar{f}_a \ln \bar{f}_a - \delta_a\left(1+\delta_a \bar{f}_a\right)\ln\left(1+\delta_a \bar{f}_a\right)\right],$$

with $\bar{f}_a = \theta_a f_a$. The qLFP kinetic equation implies

$$\frac{\partial s}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{j}_s \geq 0,$$

which expresses the local H-theorem. In particular, the total entropy $S(t) = \int d\mathbf{r}\, s(\mathbf{r}, t)$ satisfies $\frac{dS}{dt} \geq 0$ and is a monotonically increasing function of time, whatever the initial conditions.

3. Global equilibrium

As a consequence, whatever the initial conditions, the time evolution reaches a final, time-independent state, a.k.a. stationary state, when $S(t)$ reaches its maximum characterized by $\frac{dS}{dt} = 0$. The only stationary states are the Fermi-Dirac ($\delta_a = -1$) or Bose-Einstein ($\delta_a = 1$) distribution functions

$$f_a(\mathbf{r}, \mathbf{p}) = \frac{1}{\theta_a}\frac{1}{e^{-\beta\left[\mu_a - \frac{1}{2m_a}(\mathbf{p}-m_a\mathbf{u})^2\right]} - \delta_a}, \quad \forall a,$$

where the inverse temperature $\beta = 1/k_B T$, the chemical potential μ_a, and the flow velocity $\mathbf{u}$ are constant independent of $(\mathbf{r}, t)$ and are the same for all species.

4. Local thermal equilibrium

The effect of collisions vanishes only when all species are in a local Fermi-Dirac or Bose-Einstein state at the same

Reuse of AIP Publishing content is subject to the terms at: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 192.12.184.6 On: Tue, 22 Mar 2016 15:05:36

local inverse temperature $\beta(\mathbf{r},t)$ and flow velocity $\mathbf{u}(\mathbf{r},t)$. More precisely,

$$\mathcal{C}_{ab}[f_a,f_b]=0 \qquad \forall a,b$$

if and only if, $\forall a$,

$$f_a(\mathbf{r},\mathbf{p},t)=\frac{1}{\theta_a}\frac{1}{e^{-\beta(\mathbf{r},t)\left[\mu_a(\mathbf{r},t)-\frac{1}{2m_a}(\mathbf{p}-m_a\mathbf{u}(\mathbf{r},t))^2\right]}-\delta_a}. \tag{6}$$

We recall for later reference that the classical limit of the local thermal equilibrium is given by $\beta\mu_a\to-\infty$, which yields the familiar Maxwell-Boltzmann distribution

$$f_a(\mathbf{r},\mathbf{p},t)\sim n_a(\mathbf{r},t)\left(\frac{\beta}{2\pi m_a}\right)^{3/2} \times e^{-\frac{\beta}{2m_a}[\mathbf{p}_a-m_a\mathbf{u}(\mathbf{r},t)]^2} \quad \text{for} \quad \beta\mu_a\to-\infty,$$

with the local number density $n_a=\frac{1}{\theta_a}\left(\frac{2\pi m_a}{\beta}\right)^{\frac{3}{2}}e^{\beta\mu_a}$.

D. Fokker-Planck-like form of the quantum Landau-Fokker-Planck operator

The qLFP collision integral (3) can be written in the form of a non-linear Fokker-Planck collision operator[34]

$$\mathcal{C}_{ab}[f_a,f_b]=-\frac{\partial}{\partial\mathbf{p}_a}\cdot\left[\mathbf{A}_{ab}f_a(1+\delta_a\theta_af_a)+\mathbf{B}_{ab}f_a -\frac{1}{2}\frac{\partial}{\partial\mathbf{p}_a}\cdot\left(\overleftrightarrow{D}_{ab}f_a\right)\right], \tag{7}$$

$$=-\frac{\partial}{\partial\mathbf{p}_a}\cdot\left[\mathbf{A}_{ab}f_a(1+\delta_a\theta_af_a)-\frac{1}{2}\overleftrightarrow{D}_{ab}\cdot\frac{\partial}{\partial\mathbf{p}_a}f_a\right], \tag{8}$$

where we introduced the "dynamical friction" vectors

$$\mathbf{A}_{ab}(\mathbf{p}_a)=-\gamma^{ab}\int d\mathbf{p}_b\left[\frac{\partial}{\partial\mathbf{p}_b}\cdot\overleftrightarrow{V}_{ab}(\mathbf{p}_a,\mathbf{p}_b)\right]f_b(\mathbf{p}_b),$$

$$\mathbf{B}_{ab}(\mathbf{p}_a)=-\frac{m_b}{m_a}\gamma^{ab}\times\int d\mathbf{p}_b\left[\frac{\partial}{\partial\mathbf{p}_b}\cdot\overleftrightarrow{V}_{ab}(\mathbf{p}_a,\mathbf{p}_b)\right] \times f_b(\mathbf{p}_b)\left[1+\delta_b\theta_bf_b(\mathbf{p}_b)\right],$$

and the diffusion tensor

$$\overleftrightarrow{D}_{ab}(\mathbf{p}_a)=2\gamma^{ab}\int d\mathbf{p}_b\overleftrightarrow{V}_{ab}(\mathbf{p}_a,\mathbf{p}_b)f_b(\mathbf{p}_b)[1+\delta_b\theta_bf_b(\mathbf{p}_b)].$$

For simplicity, we dropped the explicit dependences on $(\mathbf{r},t)$ in the previous expressions. In deriving Eq. (8), we used the relation $\mathbf{B}_{ab}(\mathbf{p}_a)=\frac{1}{2}\frac{\partial}{\partial\mathbf{p}_a}\cdot\overleftrightarrow{D}_{ab}(\mathbf{p}_a)$.

As with the classical operator,[35] the coefficients $\mathbf{A}_{ab},\mathbf{B}_{ab}$, and $\overleftrightarrow{D}_{ab}$ can be written as

$$\mathbf{A}_{ab}(\mathbf{p}_a)=\frac{\gamma^{ab}}{m_b\mu_{ab}}\frac{\partial H_b(\mathbf{v}_a)}{\partial\mathbf{v}_a}, \tag{9a}$$

$$\mathbf{B}_{ab}(\mathbf{p}_a)=\frac{\gamma^{ab}}{m_a\mu_{ab}}\frac{\partial I_b(\mathbf{v}_a)}{\partial\mathbf{v}_a}, \tag{9b}$$

$$\overleftrightarrow{D}_{ab}(\mathbf{p}_a)=\frac{\gamma^{ab}}{\mu_{ab}}\frac{\partial^2G_b(\mathbf{v}_a)}{\partial\mathbf{v}_a\partial\mathbf{v}_a}, \tag{9c}$$

in terms of the *three* "potentials"

$$\begin{aligned} H_b(\mathbf{v})&=\int d\mathbf{v}_b\frac{\tilde{f}_b(\mathbf{v}_b)}{|\mathbf{v}-\mathbf{v}_b|},\\ I_b(\mathbf{v})&=\int d\mathbf{v}_b\frac{\tilde{f}_b(\mathbf{v}_b)\left[1+\delta_b\tilde{\theta}_b\tilde{f}_b(\mathbf{v}_b)\right]}{|\mathbf{v}-\mathbf{v}_b|},\\ G_b(\mathbf{v})&=\int d\mathbf{v}_b|\mathbf{v}-\mathbf{v}_b|\tilde{f}_b(\mathbf{v}_b)\left[1+\delta_b\tilde{\theta}_b\tilde{f}_b(\mathbf{v}_b)\right], \end{aligned} \tag{10}$$

with $\tilde{\theta}_b\equiv\theta_b/m_b^3$ and $\tilde{f}(\mathbf{v}_b)=m_b^3f(m_b\mathbf{v}_b)$. The three potentials H_b, I_b, and G_b are solution of Poisson's equations

$$\begin{aligned} \nabla^2H_b(\mathbf{v})&=-4\pi\tilde{f}_b(\mathbf{v}),\\ \nabla^2I_b(\mathbf{v})&=-4\pi\tilde{f}_b(\mathbf{v})[1+\delta_b\tilde{\theta}_b\tilde{f}_b(\mathbf{v})],\\ \nabla^2G_b(\mathbf{v})&=2I_b(\mathbf{v}), \end{aligned}$$

where $\nabla=\frac{\partial}{\partial\mathbf{v}}$.

In the case $\delta_b=0$, $\mathbf{B}_{ab}(\mathbf{p}_a)=\frac{m_b}{m_a}\mathbf{A}_{ab}(\mathbf{p}_a)$, and Eq. (7) corresponds to the usual Landau-Fokker-Planck collision operator with friction $\left(1+\frac{m_b}{m_a}\right)\mathbf{A}_{ab}$. In this case, $I_b=H_b$, and H_b and G_b reduce to the two usual Rosenbluth potentials.[35]

E. Potentials in local thermal equilibrium

We provide closed-form expressions for the potential H_b, I_b, and G_b when $\tilde{f}_b$ is a local equilibrium distribution function (6), i.e., (dropping the subscript b)

$$\tilde{f}(\mathbf{r},\mathbf{v},t)=\frac{1}{\tilde{\theta}}\frac{1}{e^{-\beta(\mathbf{r},t)\left[\mu(\mathbf{r},t)-m(\mathbf{v}-\mathbf{u}(\mathbf{r},t))^2/2\right]}-\delta}.$$

These expressions are useful in a number of applications, including the test-particle problem and linear transport problem.

We recall that, in the classical case, the equilibrium Rosenbluth potentials satisfy[17,35]

$$H(\mathbf{v})=I(\mathbf{v})=n\frac{\mathrm{erf}\left(\sqrt{\frac{\beta m}{2}}w\right)}{w}, \tag{11a}$$

$$G(\mathbf{v})=n\left[\left(w+\frac{1}{m\beta w}\right)\mathrm{erf}\left(\sqrt{\frac{\beta m}{2}}w\right)+\sqrt{\frac{2}{\pi m\beta}}e^{-\frac{\beta m}{2}w^2}\right], \tag{11b}$$

where $w=|\mathbf{v}-\mathbf{u}|$. The extension of these expressions to the quantum case is not completely trivial, and we report the lengthy details in Appendix C. The results can be conveniently expressed in terms of the usual Fermi-Dirac ($\delta=-1$) and Bose-Einstein ($\delta=1$) integrals of order ν and argument t defined as

$$\mathcal{Q}_\nu(t)=\frac{1}{\Gamma(\nu+1)}\int_0^\infty dy\frac{y^\nu}{e^{y-t}-\delta},$$

Reuse of AIP Publishing content is subject to the terms at: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 192.12.184.6 On: Tue, 22 Mar 2016 15:05:36

where $\Gamma(t)=\int_0^\infty x^{t-1}e^{-x}dx$ is the Gamma function, along with the lower incomplete integral defined as

$$\mathcal{Q}_\nu(t,x)=\frac{1}{\Gamma(\nu+1)}\int_0^x dy\frac{y^\nu}{e^{y-t}-\delta},$$

and the upper incomplete integral

$$\mathcal{Q}^c_\nu(t,x)=\mathcal{Q}_\nu(t)-\mathcal{Q}_\nu(t,x).$$

We find

$$H(\mathbf{v})=\frac{4\pi m^2}{\beta\theta}\left[\frac{\sqrt{\pi}}{2\sqrt{x}}\mathcal{Q}_{1/2}(t,x)+\mathcal{Q}^c_0(t,x)\right], \tag{12a}$$

$$I(\mathbf{v})=\frac{2m^2\pi^{3/2}}{\beta\theta\sqrt{x}}\mathcal{Q}_{-\frac{1}{2}}(t,x), \tag{12b}$$

$$G(\mathbf{v})=\frac{4\pi^{3/2}m\sqrt{x}}{\beta^2\theta}\mathcal{Q}_{-\frac{1}{2}}(t,x)+\frac{2\pi^{3/2}m}{\beta^2\theta}\frac{1}{\sqrt{x}}\mathcal{Q}_{\frac{1}{2}}(t,x)+\frac{8\pi m}{\beta^2\theta}\ln\left(1+e^{(t-x)}\right), \tag{12c}$$

where $t=\beta\mu$ and $x=\frac{\beta m w^2}{2}$. In the classical limit $\beta\mu\to-\infty$, the previous expressions reduce to the classical Rosenbluth potentials (11). This can be shown using

$$\begin{aligned}\mathcal{Q}_{\frac{1}{2}}(t,x)&\sim e^t\mathrm{erf}\left(\sqrt{x}\right)-\frac{2\sqrt{x}}{\sqrt{\pi}}e^{t-x}\\ \mathcal{Q}_{-\frac{1}{2}}(t,x)&\sim e^t\mathrm{erf}\left(\sqrt{x}\right)\\ \mathcal{Q}^c_0(t,x)&\sim e^{t-x}\end{aligned}$$

for $t\to-\infty$. In practice, the potentials can be numerically evaluated using accurate series representations of the integrals $\mathcal{Q}_\nu$, e.g., Ref. 36.

F. Scattering of a test-particle

In order to illustrate the effect of the quantum statistics on the Landau collision operator, we apply the previous results to the classic test-particle problem. We consider a classical test-particle in an otherwise homogenous electron-ion plasma at thermal equilibrium at temperature T. Electrons (e) are treated as quantum mechanical particles (with $g_a=2$ and $\delta_a=-1$ for spin 1/2 particles) and ions (i) are treated as classical particles. This composition will lead us to use and compare both the quantum and classical expressions (12) and (11) of Sec. II E. In the following, n_i and n_e denote the particle densities, $\beta_i=\beta_e=1/k_BT$ the inverse temperatures (the expressions given below apply to $\beta_e\neq\beta_i$), and $\theta=(2\pi\hbar)^3/2$; the other notations can be found in Sec. II. The test-particle constitutes a third particle species and is labeled by the letter t. We assume that the distribution function f_t of non-interacting test-particles is homogenous, so that the spatial gradient and mean-field force in the streaming operator (2) disappear, i.e., $\frac{D}{Dt}=\frac{\partial}{\partial t}$. Under these conditions, the kinetic equation (1) satisfied by f_t becomes the linear Fokker-Planck equation

$$\frac{\partial f_t}{\partial t}=-\frac{\partial}{\partial\mathbf{p}}\cdot\left[\mathbf{A}_t(\mathbf{p})f_t(\mathbf{p})-\frac{1}{2}\frac{\partial}{\partial\mathbf{p}}\cdot\left(\overleftrightarrow{D}_t(\mathbf{p})f_t(\mathbf{p})\right)\right],$$

where the dynamical friction and diffusion tensor are independent of f_t, and are given by the sum of the contributions due to collisions with electrons and ions

$$\begin{aligned}\mathbf{A}_t(\mathbf{p})&=\mathbf{A}_{te}(\mathbf{p})\left(1+\frac{m_e}{m_t}\right)+\mathbf{A}_{ti}(\mathbf{p})\left(1+\frac{m_i}{m_t}\right)\\ &\equiv-\nu_t(v)\mathbf{p}\,,\\ \overleftrightarrow{D}_t(\mathbf{p})&=\overleftrightarrow{D}_{te}(\mathbf{p})+\overleftrightarrow{D}_{ti}(\mathbf{p})\\ &\equiv m_t^2d^t_\parallel(v)\left(\overleftrightarrow{\mathrm{I}}-\frac{\mathbf{pp}}{p^2}\right)+m_t^2d^t_\perp(v)\frac{\mathbf{pp}}{p^2}.\end{aligned}$$

These contributions are calculated by applying Eq. (9) to the quantum and classical potentials (12) and (11), respectively. We obtain the friction coefficient $\nu_t=\nu_{te}+\nu_{ti}$ with

$$\begin{aligned}\nu_{te}(v)&=\frac{1}{4\pi^{\frac{3}{2}}\hbar^3}\frac{m_e^3\gamma^{te}}{m_t^2\mu_{te}}\left(1+\frac{m_t}{m_e}\right)\frac{\mathcal{Q}_{\frac{1}{2}}\left(\beta_e\mu_e,x_e^2\right)}{x_e^3},\\ \nu_{ti}(v)&=\frac{\gamma^{ti}}{m_t^2\mu_{ti}}\left(1+\frac{m_t}{m_i}\right)\frac{n_i}{v^3}\left[\mathrm{erf}(x_i)-\frac{2x_i}{\sqrt{\pi}}e^{-x_i^2}\right],\end{aligned}$$

the parallel diffusion coefficient $d^t_\parallel=d^{te}_\parallel+d^{ti}_\parallel$ with

$$d^{te}_\parallel(v)=\frac{2}{(m_t+m_e)\beta_e}\nu_{te}(v), \tag{13a}$$

$$d^{ti}_\parallel(v)=\frac{2}{(m_t+m_i)\beta_i}\nu_{ti}(v), \tag{13b}$$

and the perpendicular diffusion coefficient $d^t_\perp=d^{te}_\perp+d^{ti}_\perp$ with

$$\begin{aligned}d^{te}_\perp(v)&=\frac{1}{2\pi^{\frac{3}{2}}\hbar^3\beta_e}\frac{m_e^2\gamma^{te}}{m_t^2\mu_{te}}\\ &\quad\times\frac{1}{x_e}\left[\mathcal{Q}_{-\frac{1}{2}}\left(\beta_e\mu_e,x_e^2\right)-\frac{1}{2x_e^2}\mathcal{Q}_{\frac{1}{2}}\left(\beta_e\mu_e,x_e^2\right)\right],\\ d^{ti}_\perp(v)&=\frac{\gamma^{ti}}{m_t^2\mu_{ti}}\frac{n_i}{v}\left[\left(1-\frac{1}{2x_i^2}\right)\mathrm{erf}(x_i)+\frac{1}{\sqrt{\pi}x_i}e^{-x_i^2}\right],\end{aligned}$$

where we defined

$$x_a=\left(\frac{\beta_a m_a}{2}\right)^{1/2}v\quad\text{and}\quad v=\frac{|\mathbf{p}|}{m_t}.$$

For the illustration, we have evaluated these coefficients over a wide range of physical conditions for a proton immersed in fully ionized hydrogen (electron-proton) plasma. In the following, $v_i=\sqrt{\frac{2k_BT}{m_i}}$ (v_e) is the ion (electron) thermal velocity, $v_F=\sqrt{2E_F/m_e}$, $E_F=\frac{\hbar^2}{2m_e}(3\pi^2n_e)^{2/3}$ denote the electron Fermi velocity and Fermi energy, $\Theta=\frac{k_BT}{E_F}$ is the degeneracy parameter, which measures the degree of quantum degeneracy of electrons, $\omega_{pe}=\sqrt{\frac{4\pi e^2n_e}{m_e}}$ is the electron plasma frequency, and $a_e=\left(\frac{3}{4\pi n_e}\right)^{1/3}$ is the average distance between electrons.

Reuse of AIP Publishing content is subject to the terms at: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 192.12.184.6 On: Tue, 22 Mar 2016 15:05:36

In order to help the reader interpret physically the results, we briefly recall how the friction and diffusion coefficients are related to important processes[17] before discussing the numerical results. Under the influence of collisions with the background electrons and ions, the test-particle distribution f_t spreads out in momentum space, and ultimately becomes isotropic and Maxwellian as it reached thermal equilibrium with the electron-ion plasma. The particle's momentum $\mathbf{p}(t)$ undergoes a random walk like motion, which consists of a systematic friction force $-\nu_t\mathbf{p}(t)$ together with a random force that randomizes the direction of the momentum in directions perpendicular and parallel to the instantaneous momentum according to

$$\frac{d}{dt}\langle(\Delta p_\perp)^2\rangle = 2m_t^2 d_\perp^t, \quad \frac{d}{dt}\langle(\Delta p_\parallel)^2\rangle = m_t^2 d_\parallel^t,$$

where $\langle(\Delta p_\perp)^2\rangle$ and $\langle(\Delta p_\parallel)^2\rangle$ measure the spread of the distribution function along both directions. Finally, the rate of change of the expectation value of the test-particle's kinetic energy $W = \frac{1}{2m_t}\mathbf{p}_t^2$ is related to

$$\frac{dW}{dt} = -\nu_E W,$$

with the energy-loss rate

$$\nu_E(v) = 2\nu_t(v) - \frac{2}{v^2}\left(d_\perp^t(v) + \frac{1}{2}d_\parallel^t(v)\right),$$

with $v = |\mathbf{p}|/m_t$.

Figures 1–3 show *dimensionless* results for the slowing down rate $\nu_t(v)/\omega_{pe}$, the diffusion coefficient $d_\perp^t(v)/(a_e^2\omega_{pe}^3)$, and the energy loss rate $\nu_E(v)/\omega_{pe}$ over a wide range of conditions spanning from the classical regime ($\Theta = 50$) to the quantum degenerate regime ($\Theta = 0.01$), and for a wide range of test-particle velocities v spanning form the very slow $v \ll v_i$ to the very fast $v \gg \max(v_e, v_F)$ velocity regimes. The results shown were obtained setting the Coulomb logarithms to unity, $\ln\lambda_{te} = \ln\Lambda_{ti} = 1$, to highlight the effect of quantum statistics on the momentum integral in the Landau collision operator. The parallel diffusion coefficient is not shown since it is simply related to $\nu_t(v)$ according to Eq. (13). The arrows in Figs. 1–3 mark the location of velocities v_i, v_e, and v_F. The figures also show the separate contributions of electrons and ions on the coefficients.

The following general qualitative observations can be made regarding the effect of quantum degeneracy.

1. Friction $\nu_t(v)$

For all conditions, the slowing down rate is dominated by the ion contzribution at small enough velocity $v < v^\star$ and by the electron contribution at large enough velocity $v > v^\star$, where $v^\star/v_i$ (located by an arrow in Fig. 1) increases with decreasing Θ. While the ion thermal velocity remains for all Θ a good reference velocity that marks a net change in the collisionality with ions (see below), the reference velocity for electrons is $v_{\rm ref} = \max(v_e, v_F)$. The latter varies from the thermal velocity v_e to the Fermi velocity v_F as Θ enters quantum degenerate regime $\Theta < 1$. For energetic test-particles with $v > v^\star$, the collision with electrons become much less effective when $v > v_{\rm ref}$. Note that in the limit of full degeneracy $\Theta \to 0$, $\nu_{te}(v)$ consists of two pieces, namely,

$$\nu_{te}(v) = \frac{1}{3\pi^2\hbar^3}\frac{\gamma^{te}m_e^3}{m_t^2\mu_{te}}\left(1+\frac{m_t}{m_e}\right)\times\begin{cases}\frac{v_F^3}{v^3}, & v > v_F\\ 1, & v < v_F,\end{cases}$$

as can be seen in Fig. 1 for $\Theta = 0.01$ and 0.1. Finally, in order to see more easily the quantitative effect of the electron quantum degeneracy on the slowing-down rate, Fig. 4 (top panel) shows the ratio $\nu_t(v)/\nu_t^{\rm class}(v)$ of the results shown in Fig. 1 to the values obtained assuming classical.

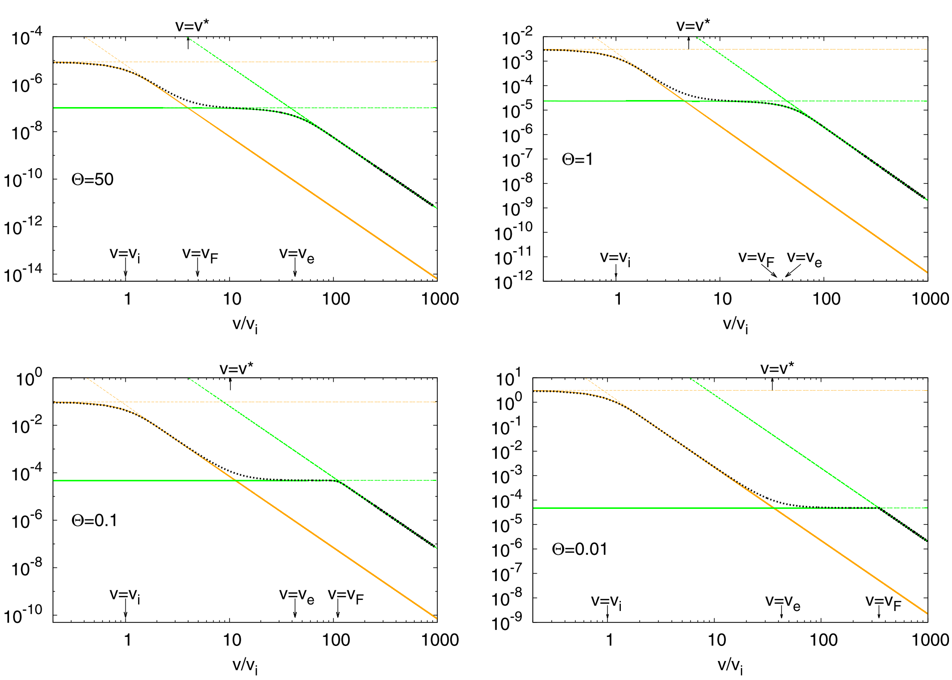

FIG. 1. Slowing-down rate $\nu_t(v)/\omega_{pe}$ (black dotted line) of a test-particle ($Z_t = 1$) of electron-ion plasmas with ion charge $Z_i = 1$, density $r_s = 1$, and electron degeneracy $\theta = 50, 1, 0.1$, and 0.01, as a function of the test-particle velocity v in units of the ion thermal velocity v_i. The arrows indicate the location of the ion thermal velocity v_i, the electron Fermi velocity v_F, the electron thermal velocity v_e, and the transition around $v^\star$ from the low velocity regime dominated by collisions with ions and the high velocity regime dominated by electron collisions. The full green line shows the contribution $\nu_{te}(v)$ of the electrons, the full orange line shows the contribution $\nu_{ti}(v)$ of the ions, while the dashed lines show the small and large test-particle velocity limits discussed in the text.

Reuse of AIP Publishing content is subject to the terms at: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 192.12.184.6 On: Tue, 22 Mar 2016 15:05:36

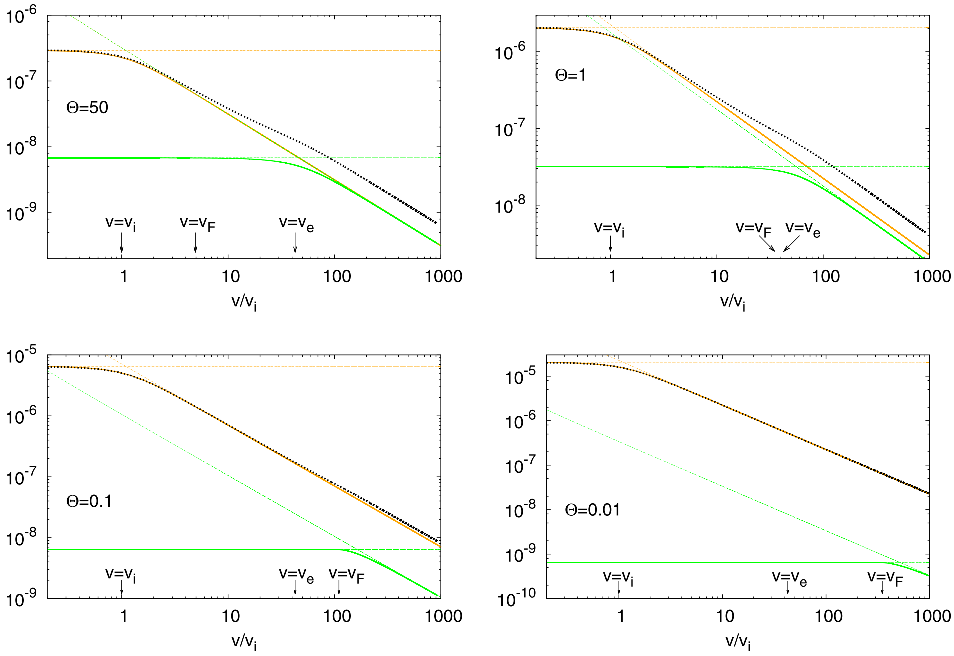

FIG. 2. Diffusion coefficient $d^t_\perp(v)/(a_e^2\omega_{pe}^3)$ of a test-particle. The conditions, notations, and legends are the same as in Fig. 1.

2. Diffusion $d_\perp(v)$

The most striking effect is the strong reduction of the electron contribution to the diffusion coefficient in the quantum degenerate regime at high velocities. While in the classical regime electrons and ions equally contribute to the perpendicular diffusion, the contribution of electrons decreases significantly when $\Theta \leq 1$. In the fully degenerate limit $\Theta \to 0$, $d^{te}_\perp(v) = 0$. To see more easily, the quantitative effect of the electron quantum degeneracy on the diffusion coefficient, Fig. 4 (bottom panel) shows the ratio $d^t_\perp(v)/d^{t,class}_\perp(v)$ of the results shown in Fig. 2 to the values obtained assuming classical electrons.

3. Energy relaxation rate $\nu_E(v)$

The energy loss rate, which is closely related to the stopping power of the electron-ion plasma, shows transitions similar to those discussed above for $\nu_t(v)$.

For practical purposes,[37] we now give explicit formulas for the three main velocity regimes readily distinguishable in Figures 1–3. These formulas give very accurate results in their range of validity; this can be seen in Figs. 1 and 2, which represent them in the low and high velocity regimes (dashes lines). Moreover, these formulas can be readily evaluated using the relation $n_e = \frac{2}{(2\pi\hbar)^3}\left(\frac{2\pi m_e}{\beta_e}\right)^{\frac{3}{2}}\mathcal{Q}_{\frac{1}{2}}(\beta_e\mu_e)$ between the particle density and $\beta_e\mu_e$, together with the simple

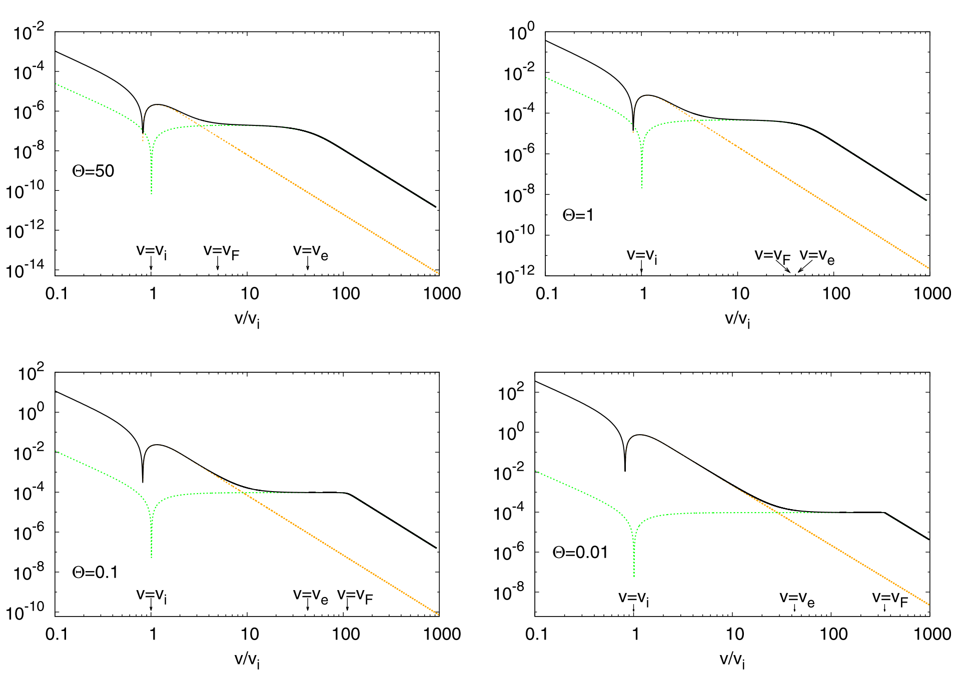

FIG. 3. Energy loss rate $|\nu_E(v)|/\omega_{pe}$ rate (black full line) of a test-particle. The conditions are the same as in Fig. 1. The energy loss $\nu_E(v)$ is positive at the right of the dip region, is zero at the minimum occurring at $v \lesssim v_i$, and is negative to its left (at small enough velocity, the test-particle absorbs energy from the plasma). The orange and green dotted lines show the contribution of the ions and electrons, respectively. The arrows indicate the location of the ion thermal velocity v_e, the electron Fermi velocity v_F and the electron thermal velocity v_e.

Reuse of AIP Publishing content is subject to the terms at: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 192.12.184.6 On: Tue, 22 Mar 2016 15:05:36

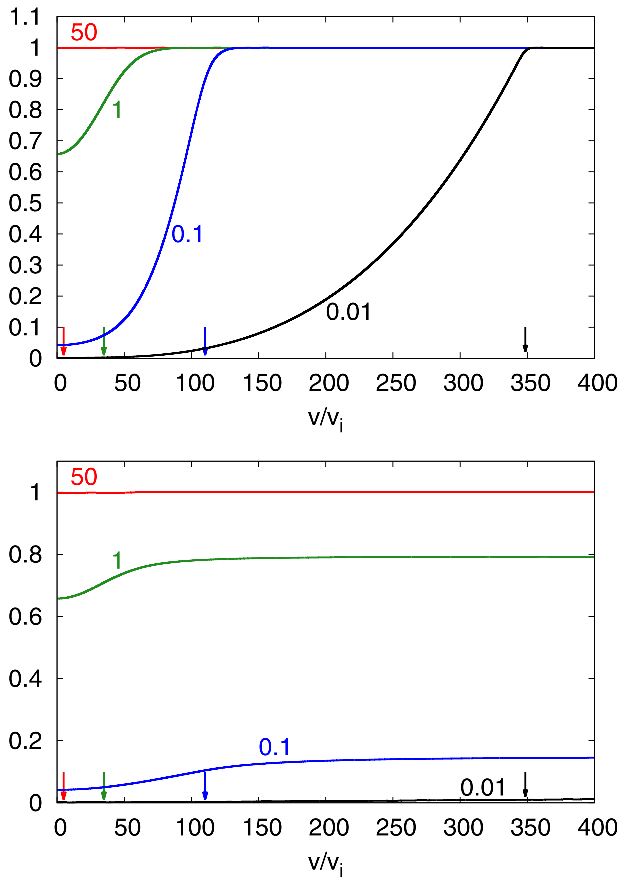

FIG. 4. Ratio of the quantum to classical slowing-down rate (top panel) and perpendicular diffusion coefficient (bottom panel) for a test-particle in an electron-ion plasma with ion charge $Z_i = 1$, density $r_s = 1$ and electron degeneracy $\Theta = 50$ (red line), 1 (green line), 0.1 (blue line), and 0.01 (black line). The colored arrows indicate the location of the Fermi velocity for each value of the degeneracy parameter.

approximate inversion formula of the Fermi integral, which can be found in Ref. 38

$$\beta_e\mu_e = -\frac{3}{2}\ln\Theta + \ln\frac{4}{3\sqrt{\pi}} + \frac{A\Theta^{-(1+b)} + B\Theta^{-(1+b)/2}}{1 + A\Theta^{-b}},$$

with $A = 0.25954$, $B = 0.0072$, and $b = 0.858$, where $\Theta = 1/(\beta_e E_F)$ is the degeneracy parameter with $E_F = \frac{\hbar^2}{2m_e}(3\pi^2 n_e)^{2/3}$.

Using the notation

$$\Gamma_t = \frac{4\pi Z_t^2 e^4}{m_t^2},$$

we have

(a) For $v \ll v_i, \max(v_e, v_F)$, the slowing-down rate and the diffusion coefficients are independent of velocity

$$\nu_t(v) = \frac{4}{3\sqrt{\pi}} n_e \Gamma_t \times \left[\frac{m_e^3}{4\pi^{\frac{3}{2}}\hbar^3 n_e}\frac{\ln\Lambda_{te}}{1 + e^{-\beta_e\mu_e}}\left(1 + \frac{m_t}{m_e}\right) + \frac{Z_i \ln\Lambda_{ti}}{v_i^3}\left(1 + \frac{m_t}{m_i}\right)\right],$$

$$d_\parallel^t(v) = \frac{4}{3\sqrt{\pi}}\Gamma_t n_e\left[\frac{m_e^3 v_e^2}{4\pi^{\frac{3}{2}}\hbar^3 n_e}\frac{\ln\Lambda_{te}}{1 + e^{-\beta_e\mu_e}} + \frac{Z_i \ln\Lambda_{ti}}{v_i}\right],$$

$$d_\perp^t(v) = d_\parallel^t(v).$$

Thus, the test-particle motion is the same as a usual Brownian motion.

(b) For $v_i \ll v \ll \max(v_e, v_F)$, the electron collisions have the same effect as in previous range, but the collisions with ions have a much different effect

$$\nu_t(v) = n_e\Gamma_t \times \left[\frac{m_e^3}{3\pi^2\hbar^3 n_e}\frac{\ln\Lambda_{te}}{1 + e^{-\beta_e\mu_e}}\left(1 + \frac{m_t}{m_e}\right) + \frac{Z_i \ln\Lambda_{ti}}{v^3}\left(1 + \frac{m_t}{m_i}\right)\right],$$

$$d_\parallel^t(v) = \Gamma_t n_e\left[\frac{m_e^3 v_e^2}{3\pi^2\hbar^3 n_e}\frac{\ln\Lambda_{te}}{1 + e^{-\beta_e\mu_e}} + \frac{Z_i \ln\Lambda_{ti} v_i^2}{v^3}\right],$$

$$d_\perp^t(v) = \Gamma_t n_e\left[\frac{m_e^3 v_e^2}{3\pi^2\hbar^3 n_e}\frac{\ln\Lambda_{te}}{1 + e^{-\beta_e\mu_e}} + \frac{Z_i \ln\Lambda_{ti}}{v}\right].$$

Diffusion due to ion collisions is primarily perpendicular to the test-particle velocity.

(c) For an energetic test-particle with $v_i, \max(v_e, v_F) \ll v$, diffusion is mainly perpendicular to the velocity of the test-particle, with electrons and ions making roughly equal contributions:

$$\nu_t(v) = \frac{n_e\Gamma_t}{v^3}\left[\left(1 + \frac{m_t}{m_e}\right)\ln\Lambda_{te} + Z_i\left(1 + \frac{m_t}{m_i}\right)\ln\Lambda_{ti}\right],$$

$$d_\parallel^t(v) = \frac{\Gamma_t n_e}{v^3}\left[v_e^2\ln\Lambda_{te} + Z_i v_i^2 \ln\Lambda_{ti}\right],$$

$$d_\perp^t(v) = \frac{\Gamma_t n_e}{v}\left[\ln\Lambda_{te} + Z_i\ln\Lambda_{ti}\right].$$

If the test-particle is an ion, the slowing-down rate is due mainly to collisions with electrons; for an electron test-particle, the electron and ion collisional contributions are roughly equal.

G. Electron-ion collisional transfer rates

Whereas electrons may, in principle, have any degree of degeneracy, the ions behave classically under most of the plasma conditions feasible in the laboratory. Accordingly, in this section, we consider the electron-ion collision operators

$$\mathcal{C}_{ei}[f_e, f_i](\mathbf{p}) = \gamma^{ei}\frac{\partial}{\partial\mathbf{p}}\cdot\int d\mathbf{p}'\,\overleftrightarrow{V}_{ei}(\mathbf{p},\mathbf{p}') \times \left\{\frac{\partial f_e(\mathbf{p})}{\partial\mathbf{p}} f_i(\mathbf{p}') - \frac{\partial f_i(\mathbf{p}')}{\partial\mathbf{p}'} f_e(\mathbf{p})[1 - \theta f_e(\mathbf{p})]\right\}$$

$$\mathcal{C}_{ie}[f_i, f_e](\mathbf{p}) = \gamma^{ei}\frac{\partial}{\partial\mathbf{p}}\cdot\int d\mathbf{p}'\,\overleftrightarrow{V}_{ie}(\mathbf{p},\mathbf{p}') \times \left\{\frac{\partial f_i(\mathbf{p})}{\partial\mathbf{p}} f_e(\mathbf{p}')[1 - \theta f_e(\mathbf{p}')] - \frac{\partial f_e(\mathbf{p}')}{\partial\mathbf{p}'} f_i(\mathbf{p})\right\}$$

Reuse of AIP Publishing content is subject to the terms at: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 192.12.184.6 On: Tue, 22 Mar 2016 15:05:36

assuming ions are classical (i.e., $(1+\delta_i\theta_i f_i)\sim 1$). Instead of focussing again on the properties of these operators as before, here we discuss and provide useful closed-form expressions for the resulting electron-ion momentum and energy transfer rates, which measure the rate at which the electron and ion subsystems exchange momentum and energy. The derivation of closed-form expressions is fairly involved, the details of which are given in Appendix D.

1. Electron-ion collisional momentum transfer rate

Dropping the dependence on $(\mathbf{r},t)$, the electron-ion momentum transfer rate, or friction force, is

$$\mathbf{F}_{ei}[f_e,f_i] = \int d\mathbf{p}\,\mathbf{p}\,C_{ei}(\mathbf{p}) = -\int d\mathbf{p}\,\mathbf{p}\,C_{ie}(\mathbf{p}) = -\mathbf{F}_{ie}[f_i,f_e].$$

That is, the collisional momentum transfer from ions to electrons is equal in magnitude and opposite in direction to that from electrons to ions, in agreement with Newton's third law. More explicitly,

$$\mathbf{F}_{ei}[f_e,f_i] = \int d\mathbf{p}\,\mathbf{A}_{ei}(\mathbf{p})f_e(\mathbf{p})\left[\left(1+\frac{m_i}{m_e}\right)-\theta_e f_e(\mathbf{p})\right],$$

with $\theta_e=(2\pi\hbar)^3/2$. The expression can be evaluated in a closed-form assuming that the ionic distribution function is a local Maxwell-Boltzmann distribution with density $n_i(\mathbf{r},t)$, mean velocity $\mathbf{u}_i(\mathbf{r},t)$, and temperature $k_BT_i(\mathbf{r},t)=1/\beta_i(\mathbf{r},t)$

$$f_i(\mathbf{r},\mathbf{p},t) = n_i\left(\frac{\beta_i}{2\pi m_i}\right)^{\frac{3}{2}} e^{-\frac{\beta_i}{2m_i}(\mathbf{p}-m_i\mathbf{u}_i)^2},$$

and that the electronic distribution function is a local Fermi-Dirac distribution with chemical potential $\mu_e(\mathbf{r},t)$, mean velocity $\mathbf{u}_e(\mathbf{r},t)$, and temperature $k_BT_e(\mathbf{r},t)=1/\beta_e(\mathbf{r},t)$

$$f_e(\mathbf{r},\mathbf{p},t) = \frac{1}{\theta_e}\frac{1}{e^{-\beta_e(\mathbf{r},t)\left[\mu_e(\mathbf{r},t)-\frac{1}{2m_e}(\mathbf{p}-m_e\mathbf{u}_e(\mathbf{r},t))^2\right]}+1}.$$

The resulting expression for $\mathbf{F}_{ei}$ is complicated (see Appendix D), but can be greatly simplified in the limit where $|\mathbf{u}_i-\mathbf{u}_e|\ll v_i$, where $v_i=\sqrt{2/m_i\beta_i}$ is the ion thermal speed.[39] After a lengthy calculation described in Appendix D, we find to first order in $\mathbf{u}_i-\mathbf{u}_e$

$$\mathbf{F}_{ei}[f_e,f_i] = -m_e n_e\frac{\mathbf{u}_e-\mathbf{u}_i}{\tau_{ei}^F},$$

where the momentum-transfer time is defined by

$$\frac{1}{\tau_{ei}^F} = \frac{8\sqrt{\pi}}{3}\frac{m_e\gamma^{ei}}{\mu_{ei}}\frac{n_i}{n_e}\frac{\beta_i}{\beta_e}\times\frac{1}{\theta_e}\int_0^\infty dy\frac{e^{-y^2}}{1+e^{-\beta_e\mu_e}e^{\frac{m_e\beta_e}{m_i\beta_i}y^2}}\left[2\left(\frac{\beta_e}{\beta_i}-1\right)y^2+1\right]. \quad (14)$$

In the classical limit $\beta_e\mu_e\to-\infty$, the expression (14) reduces to the usual result[17,40]

$$\frac{1}{\tau_{ei}^F}\sim\frac{4}{3\sqrt{\pi}}\frac{\gamma^{ei}}{m_e^2\mu_{ei}}\frac{n_i}{\left(v_e^2+v_i^2\right)^{3/2}}\left(1+\frac{m_e}{m_i}\right), \quad (15)$$

with $v_e=\sqrt{2/m_e\beta_e}$. In many physical situations, $\frac{m_e\beta_e}{m_i\beta_i}\ll 1$, and Eq. (14) is then well approximated by

$$\frac{1}{\tau_{ei}^F}\sim\frac{1}{\tau_{ei}}\equiv\frac{4\pi}{3}\frac{n_i}{n_e}\frac{\gamma^{ei}}{\theta_e}\frac{1}{1+e^{-\beta_e\mu_e}} \quad \text{for} \quad \frac{m_e\beta_e}{m_i\beta_i}\ll 1. \quad (16)$$

Figure 5 shows the relaxation time τ_{ei} as a function of the degeneracy parameter Θ.

2. Electron-ion collisional energy transfer rate

The electron-ion collisional energy exchange rate is

$$Q_{ei}[f_e,f_i] = \int d\mathbf{p}\frac{(\mathbf{p}-m_e\mathbf{u}_e)^2}{2m_e}C_{ei}(\mathbf{p}) = \frac{1}{m_e}\int d\mathbf{p}\left\{(\mathbf{p}-m_e\mathbf{u}_e)\cdot\mathbf{A}_{ei}(\mathbf{p})f_e(\mathbf{p})\times\left[\left(1+\frac{m_i}{m_e}\right)-\theta f_e(\mathbf{p})\right]+\frac{1}{2}\mathrm{Tr}\overleftrightarrow{D}_{ei}(\mathbf{p})f_e(\mathbf{p})\right\}.$$

It is related to the ion-electron rate such as

$$Q_{ei}+Q_{ie} = (\mathbf{u}_i-\mathbf{u}_e)\cdot\mathbf{F}_{ei},$$

which expresses the conservation of energy in collisions between electrons and ions. With the local distribution functions considered in Sec. II G 1, we find, assuming $||\mathbf{u}_i-\mathbf{u}_e||/v_i\ll 1$,

$$Q_{ei}[f_e,f_i] = 3k_Bn_e\left(1+\frac{m_i}{m_e}\right)^{-1}\frac{T_i-T_e}{\tau_{ei}^Q} = -Q_{ie}[f_i,f_e],$$

where the energy relaxation time is

$$\frac{1}{\tau_{ei}^Q} = \frac{16\sqrt{\pi}}{3}\frac{m_e\gamma^{ei}}{\mu_{ei}}\frac{n_i}{n_e}\left(1+\frac{m_e}{m_i}\right)\times\frac{1}{\theta_e}\int_0^\infty dy\frac{y^2e^{-y^2}}{1+e^{-\beta_e\mu_e}e^{\frac{m_e\beta_e}{m_i\beta_i}y^2}}. \quad (17)$$

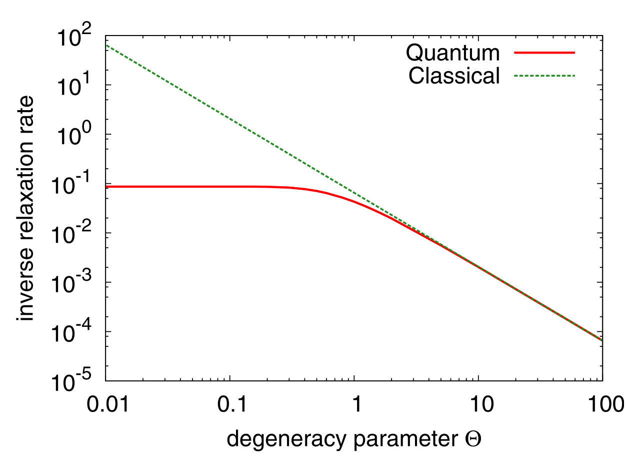

FIG. 5. Inverse relaxation rate ω_{pe}/τ_{ei}, Eq. (16), and its classical limit (15) as a function of the degeneracy parameter Θ in an hydrogen plasma with $n_e=1.28\times10^{25}\,\mathrm{cm}^{-3}$.

Reuse of AIP Publishing content is subject to the terms at: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 192.12.184.6 On: Tue, 22 Mar 2016 15:05:36

In the classical limit $\beta_e\mu_e \to -\infty$, the expression (17) reduces to the usual result[17]

$$\frac{1}{\tau_{ei}^{Q}} \sim \frac{4}{3\sqrt{\pi}}\frac{\gamma^{ei}}{m_e^2\mu_{ei}}\frac{n_i}{\left(v_e^2+v_i^2\right)^{3/2}}\left(1+\frac{m_e}{m_i}\right).$$

In the limit $\frac{m_e\beta_e}{m_i\beta_i} \ll 1$, Eq. (17) is well approximated by

$$\frac{1}{\tau_{ei}^{Q}} \sim \frac{1}{\tau_{ei}},$$

where τ_{ei} is defined as in Eq. (16). This result corresponds to the popular result of Brysk *et al.*[30] that was obtained by extending the usual binary Coulomb collision calculation of Spitzer[40] to include the Pauli principle. Further discussion on τ_{ei} in dense plasmas can be found in Ref. 41.

III. LINEARIZED QUANTUM LANDAU COLLISION OPERATOR: ELECTRON-ELECTRON COLLISIONS

The non-linearity of a collision operator is essential if the state of the system is far from local thermal equilibrium. However, in the important situations where the phase-space distribution f remains near local thermal equilibrium f_0, the linearized form of the collision operator provides an accurate description of the dynamics of the deviation $\delta f = f - f_0$, while the dynamics of f_0 is governed the hydrodynamic equations through its dependence on the thermodynamic variables. More generally, linearized collision operators play an important role in the mathematical analysis of kinetic equations based on perturbation expansions, such as in the celebrated Chapman-Enskog method. Moreover, linearized collision operators are at the basis of advanced numerical algorithms to include the effect of collisions in kinetic simulations (e.g., the δf-method in traditional plasma physics[21,42,43]). The extension of such algorithms to the qLFP operator could be used in the applications to dense plasmas briefly mentioned in Sec. II. In this section, we discuss the properties of the operator obtained by linearizing the qLFP operator around local equilibrium. First we describe general properties in connection with the conservation laws, the stationary states, and the self-adjointness and positivity of the linerarized operator. From these properties, the well-known Chapman-Enskog solution of the classical Boltzmann equation[20] can be straightforwardly adapted to the qLFP operator. Then we discuss two approximations of the linearized qLFP operator that can be useful in numerical implementations of the latter for modeling dense plasmas near local equilibrium.

We focus on the linearization of the operator $\mathcal{C}_{aa}$ for like-particle collisions; the extension to unlike-particle collision operator $\mathcal{C}_{ab}$ is straightforward. For definiteness, but without lack of generality, we consider the electron-electron collision operator (setting $\delta_a = -1$, $g_a = 2$ in the $\mathcal{C}_{aa}$). For simplicity of notation, we drop the subscript "e" in most expressions.

A. Generalities

We assume that at every space-time position $(\mathbf{r}, t)$, the momentum distribution function can be decomposed as

$$f = f_0 + \delta f,$$

where

$$f_0(\mathbf{r},\mathbf{p},t) = \frac{2}{(2\pi\hbar)^3}\frac{1}{e^{-\beta(\mathbf{r},t)\left[\mu(\mathbf{r},t)-\frac{1}{2m}(\mathbf{p}-m\mathbf{u}(\mathbf{r},t))^2\right]}+1}$$

is the local Fermi-Dirac distribution function and

$$\delta f \ll f.$$

For convenience we define the momentum in the reference frame

$$\mathbf{g}(\mathbf{r},\mathbf{p},t) = \mathbf{p} - m\mathbf{u}(\mathbf{r},t), \quad g = |\mathbf{g}|,$$

and the function $F_0(\mathbf{g}) = f_0(\mathbf{g} + m\mathbf{u})$.

Expanding the electron-electron qLFP collision operator to first order in δf, we obtain

$$\mathcal{C}_{ee}[f,f] = \underbrace{\mathcal{C}_{ee}[f_0,f_0]}_{=0} + \hat{\mathcal{C}}\delta f + O(\delta f^2),$$

where

$$\hat{\mathcal{C}}\delta f = \mathcal{C}_1[f_0,\delta f] + \mathcal{C}_2[\delta f, f_0]$$

is the linearized qLFP collision operator.

1. First term

$\mathcal{C}_1$ is a differential operator acting on δf, more precisely a linear Fokker-Planck operator

$$\mathcal{C}_1[f_0,\delta f] = -\frac{\partial}{\partial \mathbf{p}}\cdot\left[\mathbf{C}\,\delta f - \frac{1}{2}\frac{\partial}{\partial \mathbf{p}}\cdot\left(\overleftrightarrow{D}\delta f\right)\right], \tag{18}$$

where the friction vector $\mathbf{C}$ and diffusion tensor $\overleftrightarrow{D}$ are independent of δf and are given by

$$\mathbf{C} = (1-2\theta f_0)\mathbf{A}_{ee}[f_0] + \mathbf{B}_{ee}[f_0], \tag{19}$$

$$\overleftrightarrow{D} = \overleftrightarrow{D}_{ee}[f_0] \tag{20}$$

in terms of the friction vectors and diffusion tensor defined in Eq. (9). Using the expressions (12) for the potentials in local equilibrium into Eq. (9), we obtain the following closed-form expression:

$$\mathbf{C}(\mathbf{r},\mathbf{p},t) = c(\mathbf{r},\mathbf{g},t)\mathbf{g},$$
$$\overleftrightarrow{D}(\mathbf{r},\mathbf{p},t) = d_{\parallel}(\mathbf{r},\mathbf{g},t)\frac{\mathbf{g}\mathbf{g}}{g^2} + d_{\perp}(\mathbf{r},\mathbf{g},t)\left(\overleftrightarrow{\mathrm{I}} - \frac{\mathbf{g}\mathbf{g}}{g^2}\right),$$

where

$$c(\mathbf{g}) = a(\mathbf{g})(1-2\theta F_0(\mathbf{g})) + b(\mathbf{g})$$

and

$$a(\mathbf{g}) = -\frac{4\gamma^{ee}}{g^3}\left(\frac{m}{2\pi\hbar^2\beta}\right)^{\frac{3}{2}}\mathcal{Q}_{\frac{1}{2}}(\beta\mu, x), \tag{21a}$$

Reuse of AIP Publishing content is subject to the terms at: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 192.12.184.6 On: Tue, 22 Mar 2016 15:05:36

$$b(\mathbf{g}) = -\frac{4\gamma^{ee}}{g^3}\left(\frac{m}{2\pi\hbar^2\beta}\right)^{\frac{3}{2}}\mathcal{Q}_{-\frac{1}{2}}(\beta\mu,x) + \frac{8\pi\gamma^{ee}m}{\beta g^2}F_0(\mathbf{g}),$$

$$d_{\parallel}(\mathbf{g}) = -\frac{2m}{\beta}a(\mathbf{g}), \tag{21b}$$

$$d_{\perp}(\mathbf{g}) = \frac{4\gamma^{ee}}{g}\left(\frac{m}{2\pi\hbar^2\beta}\right)^{\frac{3}{2}}\mathcal{Q}_{-\frac{1}{2}}(\beta\mu,x) - \frac{1}{2}d_{\parallel}(\mathbf{g}), \tag{21c}$$

where $x = \frac{\beta}{2m}g^2$.

In the classical limit, $\mathcal{C}_1$ reduces to the collision (Fokker-Planck) operator of a test-particle colliding with a Maxwellian background.[17] In contrast, the general expression (18) differs from that of a test-particle moving in the equilibrium, Fermi-Dirac electronic background (as previously discussed in Sec. II E). Nevertheless, $\mathcal{C}_1$ can still be regarded as a drag-diffusion operator in momentum space, with drag coefficient c and diffusion coefficients $d_{\parallel}$ and $d_{\perp}$.

2. Second term

The second term

$$\mathcal{C}_2[\delta f, f_0] = -\frac{\partial}{\partial \mathbf{p}}\cdot\left\{\delta\mathbf{C}[\delta f]f_0 - \frac{1}{2}\frac{\partial}{\partial \mathbf{p}}\cdot\left[\overleftrightarrow{\delta D}[\delta f]f_0\right]\right\},$$

where

$$\delta\mathbf{C}[\delta f] = (1-\theta f_0)\mathbf{A}_{ee}[\delta f] + \mathbf{B}_{ee}[(1-2\theta f_0)\delta f],$$
$$\overleftrightarrow{\delta D}[\delta f] = \overleftrightarrow{D}_{ee}[(1-2\theta f_0)\delta f].$$

The term $\mathcal{C}_2$ consists of source and sink terms that enforce the conservation laws of the full operator $\hat{\mathcal{C}}$. While simpler than the non-linear operator $\mathcal{C}_{ee}$, $\mathcal{C}_2$ is nevertheless still complicated to deal with analytically and numerically since it is a non-local integral operator of the form

$$\mathcal{C}_2[\delta f, f_0](\mathbf{p}) = \int d\mathbf{p}' K(\mathbf{p},\mathbf{p}')\delta f(\mathbf{p}'). \tag{22}$$

Below we discuss approximations of $\mathcal{C}_2$ that can be used to facilitate its treatment in practical applications.

3. Alternative expression

For some applications, the following expression of the linearized operator $\hat{\mathcal{C}}$ can be useful

$$\hat{\mathcal{C}}\delta f = \gamma^{ee}\frac{\partial}{\partial \mathbf{p}}\cdot\int d\mathbf{p}'\,\overleftrightarrow{V}(\mathbf{p},\mathbf{p}')\cdot\left[\frac{\partial\phi(\mathbf{p})}{\partial\mathbf{p}} - \frac{\partial\phi(\mathbf{p}')}{\partial\mathbf{p}'}\right].$$

where $\delta f = f_0(1-\theta f_0)\phi$.

B. Properties

The linearized operator $\hat{\mathcal{C}}$ has most of the same properties as the non-linear collision term $\mathcal{C}_{ee}$ (see Sec. II C) and as its classical counterpart.[17] The properties listed here can be important in analytical works and numerical applications. Interestingly, several of them are satisfied separately by the terms $\mathcal{C}_1$ and $\mathcal{C}_2$.

1. Collisional invariants

The quantities $(1, \mathbf{p}, \mathbf{p}^2)$ are the collisional invariants of $\hat{\mathcal{C}}$, i.e.,

$$\int d\mathbf{p}\,\psi\hat{\mathcal{C}}\delta f = 0 \quad \text{for} \quad \psi = 1, p_x, p_y, p_z, \mathbf{p}^2.$$

2. Self-adjointness

This important property of the linearized collision operator is arguably more difficult to prove than in the classical case. The details of the proof are given in Appendix E 1.

Let $\delta f = f_0(1-\theta f_0)a$, where a is a scalar function of momentum $\mathbf{p}$; we define

$$I_0(a) = \hat{\mathcal{C}}\delta f = I_1(a) + I_2(a),$$

with

$$I_1(a) = \mathcal{C}_1[f_0, \delta f], \quad I_2(a) = \mathcal{C}_2[\delta f, f_0].$$

Given two functions a and b of the momentum $\mathbf{p}$, we define the bracket integrals

$$[a,b]_n = \int d\mathbf{p}\, bI_n(a) \quad \text{with } n = 0,1,2.$$

The following properties are satisfied:

(a) the bracket integrals are bilinear, symmetric forms, i.e.,

$$[a,b]_n = [b,a]_n \quad \text{with } n = 0,1,2. \tag{23}$$

(b) I_0 is a semi-definite positive operator in the sense that, for arbitrary a

$$[a,a]_0 \geq 0.$$

The equality sign holds if and only if a is a linear combination of the collisional invariants

$$a(\mathbf{p}) = c_0 + \mathbf{c}_1\cdot\mathbf{p} + c_2\mathbf{p}^2, \tag{24}$$

where c_0, $\mathbf{c}_1$, and c_2 are independent of $\mathbf{p}$;

(c) consequently, the general solution of the homogeneous integral equation $I_0(a) = 0$ is given by Eq. (24), i.e.,

$$\hat{\mathcal{C}}\delta f = 0 \iff \delta f = (c_0 + \mathbf{c}_1\cdot\mathbf{p} + c_2\mathbf{p}^2)f_0(1-\theta f_0). \tag{25}$$

Physically, Eq. (25) can be regarded as the general expression for the modification of a local Fermi-Dirac distribution function due to perturbations in the thermodynamic variables μ, β, and $\mathbf{u}$, Taylor expanded to first order in these perturbation. Indeed, by substituting $\mu + \delta\mu$ for μ (and similarly for β and $\mathbf{u}$) in Eq. (18), and Taylor expanding with respect to the variations $\delta\mu$, $\delta\beta$ and $\delta\mathbf{u}$, the first order term is

$$\delta f = \left[\beta\delta\mu + \left(\mu - \frac{(\mathbf{p}-m\mathbf{u})^2}{2m}\right)\delta\beta + \beta(\mathbf{p}-m\mathbf{u})\cdot\delta\mathbf{u}\right] \times f_0(1-\theta f_0).$$

Reuse of AIP Publishing content is subject to the terms at: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 192.12.184.6 On: Tue, 22 Mar 2016 15:05:36

C. Approximations

As mentioned above, in applications, rather than use the complicated integral operator $\mathcal{C}_2$, it may be more convenient to employ a simpler approximate operator that shares as many properties as possible with the exact operator. At the least, to be physically acceptable, one should replace $\mathcal{C}_2$ by a term that guarantees local particle number, momentum, and energy conservation such that the particles, momentum, and energy removed by the drag-diffusion term $\mathcal{C}_1$ is replenished by the approximate $\mathcal{C}_2$.

In the following, we generalize two approximations of the classical, linearized Fokker-Planck operator commonly used in traditional plasma physics.[42–45] We begin by extending to the quantum case the approximation introduced by Catto and Tsang and later by other authors,[42,44,45] and then we consider the refined formulation of Lin, Tang, and Lee.[43] For convenience, we remark that both approximations can be written as

$$\mathcal{C}_2[\delta f, f_0] \approx f_0(1-\theta f_0)\mathcal{O}[\delta f],$$

with

$$\begin{aligned}\mathcal{O}[\delta f] &= -\mathbf{K}\cdot\int d\mathbf{p}'\,[\mathbf{p}'-m\mathbf{u}]\mathcal{C}_1[f_0,\delta f'] \\ &\quad -\mathcal{E}\int d\mathbf{p}'\,[a(\mathbf{p}'-m\mathbf{u})^2+b]\mathcal{C}_1[f_0,\delta f'] \qquad (26a)\\ &= -\mathbf{K}\cdot\int d\mathbf{p}'\,\mathbf{C}\delta f' \\ &\quad -\mathcal{E}\int d\mathbf{p}'\,a[2\mathbf{C}\cdot(\mathbf{p}'-m\mathbf{u})+\mathrm{Tr}\overleftrightarrow{D}]\delta f', \qquad (26b)\end{aligned}$$

where $\delta f' = \delta f(\mathbf{r},\mathbf{p}',t)$ (note that in deriving the last equation (26b), b was assumed to be independent of $\mathbf{p}'$).

Comparing Eqs. (26) with (22), one sees that the former is significantly simpler to evaluate for different values of the momentum $\mathbf{p}$ than the exact operator; in particular, the momentum integrals of δf in $\mathcal{O}[\delta f](\mathbf{r},\mathbf{p},t)$ is the same for all values of $\mathbf{p}$ and, in contrast to Eq. (22), need to be evaluated only once at each space-time point $(\mathbf{r},t)$.

1. First approximation

Although not explicitly mentioned in the original papers, the approximation of Refs. 44 and 45 is obtained by expanding the classical limit $\mathcal{C}_2[\delta f, f_0]$ over orthogonal trivariate polynomials with respect to the local Maxwellian distribution function, e.g., the Hermite tensor polynomials introduced by Grad,[46] and then keeping only the terms that are strictly necessary to ensure the conservation of particle number, momentum, and energy, and setting all the other terms to zero. The generalization to the quantum case requires polynomials orthogonal with respect to $f_0(1-\theta f_0)$, which leads to Eq. (26) with

$$a = \frac{\beta}{m}, \quad b = -3\frac{\mathcal{Q}_{\frac{1}{2}}(\beta\mu)}{\mathcal{Q}_{-\frac{1}{2}}(\beta\mu)}.$$

One can easily verify[47,48] that the polynomials $H(\mathbf{g})=1$, $g_i\,(i=1,2,3)$, and $a\mathbf{g}^2+b$ are, indeed, orthogonal with respect to $f_0(1-\theta f_0)$. Enforcing the constraints of conservation of particle number, momentum and energy, we obtain the following expressions for $\mathbf{K}$ and $\mathcal{E}$:

$$\mathbf{K}(\mathbf{r},\mathbf{p},t) = \alpha_K(\mathbf{r},t)\mathbf{g},$$

with

$$\begin{aligned}\alpha_K &= \left[\frac{1}{3}\int d\mathbf{p}\,g^2 f_0(1-\theta f_0)\right]^{-1} \\ &= \left[nmk_BT\frac{\mathcal{Q}_{\frac{1}{2}}(\beta\mu)}{\mathcal{Q}_{-\frac{1}{2}}(\beta\mu)}\right]^{-1}\end{aligned}$$

and

$$\mathcal{E}(\mathbf{r},\mathbf{p},t) = \alpha_{\mathcal{E}}(\mathbf{r},t)\left(\beta\frac{(\mathbf{p}-m\mathbf{u}(\mathbf{r},t))^2}{m} - 3\frac{\mathcal{Q}_{\frac{1}{2}}(\beta\mu)}{\mathcal{Q}_{-\frac{1}{2}}(\beta\mu)}\right),$$

with

$$\begin{aligned}\alpha_{\mathcal{E}} &= \left[\int d\mathbf{p}\left(\frac{\beta}{m}g^2 - 3\frac{\mathcal{Q}_{\frac{1}{2}}(\beta\mu)}{\mathcal{Q}_{-\frac{1}{2}}(\beta\mu)}\right)^2 f_0(1-\theta f_0)\right]^{-1} \\ &= \left[15n\frac{\mathcal{Q}_{\frac{3}{2}}(\beta\mu)}{\mathcal{Q}_{-\frac{1}{2}}(\beta\mu)} - 9n\left(\frac{\mathcal{Q}_{\frac{1}{2}}(\beta\mu)}{\mathcal{Q}_{-\frac{1}{2}}(\beta\mu)}\right)^2\right]^{-1}.\end{aligned}$$

In the classical limit $\beta\mu\to-\infty$ obtained using $Q_\nu(t)\sim 1$ for $t\to-\infty$, the previous expressions give

$$\alpha_K = \frac{1}{mnk_BT}, \quad \alpha_{\mathcal{E}} = \frac{1}{6n},$$

which correspond to the usual values used in the literature.[44,45]

2. Second approximation

This approximation improves the first approximation in that, like the exact operator, it annihilates functions δf of the form (25). The approximation corresponds to setting

$$a = 1, \quad b = 0,$$

in Eq. (26) together with

$$\mathbf{K}(\mathbf{r},\mathbf{p},t) = \alpha_K(\mathbf{r},t)\mathbf{C}(\mathbf{r},\mathbf{p},t) = \alpha_K(\mathbf{r},t)c(\mathbf{r},\mathbf{g},t)\mathbf{g},$$

with

$$\begin{aligned}\alpha_K &= \left[\frac{1}{3}\int d\mathbf{p}\,cg^2 f_0(1-\theta f_0)\right]^{-1} \\ &= \left[\frac{4\gamma^{ee}}{3\theta^2}\left(\frac{2\pi m}{\beta}\right)^{\frac{5}{2}}\left(\mathcal{Q}_{-\frac{3}{2}}(\beta\mu) - \mathcal{Q}_{-\frac{1}{2}}(\beta\mu)\right)\right]^{-1} \qquad (27)\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}(\mathbf{r},\mathbf{p},t) &= \alpha_{\mathcal{E}}(\mathbf{r},t)[2\mathbf{C}(\mathbf{r},\mathbf{p},t)\cdot\mathbf{g} + \mathrm{Tr}\overleftrightarrow{D}(\mathbf{r},\mathbf{p},t)] \\ &= \alpha_{\mathcal{E}}(\mathbf{r},t)[2c(\mathbf{r},\mathbf{g},t)g^2 + d_\parallel(\mathbf{r},\mathbf{g},t) + 2d_\perp(\mathbf{r},\mathbf{g},t)],\end{aligned}$$

with

Reuse of AIP Publishing content is subject to the terms at: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 192.12.184.6 On: Tue, 22 Mar 2016 15:05:36

$$\alpha_{\mathcal{E}} = \left[\int d\mathbf{p}\left[2cg^4 + \left(d_{\parallel} + 2d_{\perp}\right)g^2\right] f_0(1-\theta f_0)\right]^{-1}$$
$$= \left[\frac{128\,\pi^{5/2}\gamma^{ee}}{\theta^2}\left(\frac{m}{\beta}\right)^{\frac{7}{2}}\left(\mathcal{Q}_{-\frac{1}{2}}(\beta\mu) - \mathcal{Q}_{\frac{1}{2}}(\beta\mu)\right)\right]^{-1}.$$

In the classical limit (obtained using the series expansion $\mathcal{F}_\nu(t) = z - \frac{z^2}{2^{\nu+1}} + o(z^2)$ with $z = e^t$), the previous expressions give

$$\alpha_K = \left(-\frac{2n^2\gamma^{ee}}{3}\sqrt{\frac{\beta}{m\pi}}\right)^{-1}, \quad \alpha_{\mathcal{E}} = \left(-4n^2\gamma^{ee}\sqrt{\frac{m}{\pi\beta}}\right)^{-1},$$

which correspond to those originally proposed by Lin *et al.*[21,43]

This approximation satisfies many properties of the exact operator previously discussed in Sec. III B.

3. Self-adjointness

Let $\delta f = f_0(1-\theta f_0)a$, and

$$I_{\tilde{2}}[a] \equiv f_0(1-\theta f_0)\hat{\mathcal{O}}\delta f, \quad [a,b]_{\tilde{2}} = \int d\mathbf{p}\, b I_{\tilde{2}}[a].$$

Then, by construction, as shown in Appendix E 2

$$[a,b]_{\tilde{2}} = [b,a]_{\tilde{2}}. \tag{28}$$

As a consequence, the approximate linearized collision operator

$$\tilde{\mathcal{C}}\delta f = \mathcal{C}_1(f_0,\delta f) + f_0(1-\theta f_0)\mathcal{O}[\delta f]$$

is also self-adjoint.

4. Conservation laws

By construction, the quantities $(1, \mathbf{p}, \mathbf{p}^2)$ are the collision invariants of $\tilde{\mathcal{C}}$, i.e.,

$$\int d\mathbf{p}(c_0 + \mathbf{c}_1\cdot\mathbf{p} + c_2\mathbf{p}^2)\tilde{\mathcal{C}}\delta f = 0, \quad \forall \delta f, \tag{29}$$

where c_0, $\mathbf{c}_1$, and c_2 are independent of $\mathbf{p}$.

5. Stationary states

$\tilde{\mathcal{C}}$ satisfies[49]

$$\tilde{\mathcal{C}}\delta f = 0 \iff \delta f = (c_0 + \mathbf{c}_1\cdot\mathbf{p} + c_2\mathbf{p}^2)f_0(1-\theta f_0).$$

Like the exact linearized operator $\hat{\mathcal{C}}$ (see Eq. (25), but unlike the first approximation, the second approximation annihilates the collisional steady states, i.e., the linearly shifted Fermi-Dirac distribution functions. This is because by taking into account the momentum dependence of the momentum and energy exchange rates induced by collisions, the second approximation maintains a linearly shifted Fermi-Dirac distribution by restoring the momentum and energy according to their loss rates. In contrast, in the first approximation, an initially linearly shifted Fermi-Dirac distribution function is being distorted in momentum space over time.

IV. CONCLUSION

We have extended many of the standard properties of the classical Landau-Fokker-Plank collision operator widely used in plasma physics to the quantum Landau collision operator, which extends the former operator to include effects of quantum statistics. First, we have discussed general aspects of the qLFP operator, including properties in connection with the conservation laws, the H-theorem, and the global and local equilibrium distributions; its Fokker-Planck form in terms of three potentials that extend the usual two Rosenbluth potentials; the establishment of useful closed-form expressions for these potentials in terms of Fermi-Dirac and Bose-Einstein integrals; the application of the latter to the classic test-particle problem to illustrate the physics embodied by the qLFP operator; the development of useful closed-form expressions for the electron-ion momentum and energy transfer rates. Then, we have discussed the basic properties of the linearized qLFP operator, and extended two classic approximations of its classical counterpart that can be useful in numerical implementations. The algebraic manipulations needed in establishing useful, closed-form expressions are arguably less straightforward than in the classical case. We have therefore given all the derivations in the appendixes not only for completeness but also because the "tricks" used could potentially be useful to other quantum kinetic theory calculations.

ACKNOWLEDGMENTS

This work was performed under the auspices of the United States Department of Energy under Contract No. DE-AC52-06NA25396. This research was supported by the DOE Office of Fusion Energy Sciences, and by an LDRD grant at Los Alamos National Laboratory.

APPENDIX A: A DERIVATION OF THE QUANTUM LANDAU COLLISION OPERATOR

We present a physicist's derivation of the qLFP collision operator. For simplicity of notation, we consider an electron plasma in a uniform positive charge background; in this appendix, m is the electron mass, $u(k) = 4\pi e^2/k^2$ is Fourier transform of the bare Coulomb potential energy e^2/r of two electrons at a distance r apart. We start from the quantum Boltzmann (qB) collision integral for the rate of change of the number of electrons in momentum state $\mathbf{p}$

$$\mathcal{C}^{qB}[f,f](\mathbf{p}) = \int d\mathbf{p}'\int \frac{d\mathbf{q}}{(2\pi\hbar)^3}\left|\frac{v(q/\hbar)}{\epsilon\left(q/\hbar, \frac{\mathbf{q}\cdot\mathbf{p}}{\hbar m} + \frac{q^2}{2m}\right)}\right|^2 \times \frac{2\pi m}{\hbar}\delta\left(\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}'+\mathbf{q})\right) \times \left[f_{\mathbf{p}+\mathbf{q}}f_{\mathbf{p}'-\mathbf{q}}\left(1-\theta f_{\mathbf{p}}\right)\left(1-\theta f_{\mathbf{p}'}\right) - f_{\mathbf{p}}f_{\mathbf{p}'}\left(1-\theta f_{\mathbf{p}+\mathbf{q}}\right)\left(1-\theta f_{\mathbf{p}'-\mathbf{q}}\right)\right], \tag{A1}$$

Reuse of AIP Publishing content is subject to the terms at: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 192.12.184.6 On: Tue, 22 Mar 2016 15:05:36

where the transition probability per unit time for Coulomb scattering of two electrons from momentum state $\mathbf{p}, \mathbf{p}'$ to momentum states $\mathbf{p}+\mathbf{q}$, $\mathbf{p}-\mathbf{q}$ accounts for the screening effect via the dielectric function $\epsilon(k,\omega)$.[29]

The qLFP collision integral is obtained by retaining in $\mathcal{C}^{qB}$ only the small angle scattering events. This is done by expanding the integrand in powers of the momentum transfer $\mathbf{q}$ and keeping only the leading term. While the calculation does not present any major difficulty, the bookkeeping of terms of the same order requires some attention in order to reduce to the compact form Eq. (3). Below we outline the main steps.

We combine the expansion to first-order in $\mathbf{q}$ of both the delta function

$$\delta\big(\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}'+\mathbf{q})\big) \approx \delta\big(\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')\big) + \mathbf{q}\cdot\frac{\partial}{\partial\mathbf{p}}\delta\big(\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')\big)$$

and the dielectric function

$$\left|\frac{v(q/\hbar)}{\epsilon\left(q/\hbar, \dfrac{\mathbf{q}\cdot\mathbf{p}}{\hbar m}+\dfrac{q^2}{2m}\right)}\right|^2 \approx \left|\frac{v(q/\hbar)}{\epsilon\left(q/\hbar, \dfrac{\mathbf{q}\cdot\mathbf{p}}{\hbar m}\right)}\right|^2 + \frac{\mathbf{q}}{2}\cdot\frac{\partial}{\partial\mathbf{p}}\left|\frac{v(q/\hbar)}{\epsilon\left(q/\hbar, \dfrac{\mathbf{q}\cdot\mathbf{p}}{\hbar m}\right)}\right|^2,$$

with the Taylor expansion to second order in $\mathbf{q}$ of the term in brackets in Eq. (A1)

$$\begin{aligned}[\ldots] \approx &\left[\mathbf{q}\cdot\frac{\partial f_{\mathbf{p}}}{\partial\mathbf{p}}f_{\mathbf{p}'}(1-\theta f_{\mathbf{p}'}) - \mathbf{q}\cdot\frac{\partial f_{\mathbf{p}'}}{\partial\mathbf{p}'}(1-\theta f_{\mathbf{p}})f_{\mathbf{p}}\right] + \left[\frac{1}{2}\mathbf{q}\cdot Df_{\mathbf{p}}\cdot\mathbf{q}f_{\mathbf{p}'}(1-\theta f_{\mathbf{p}'}) + \frac{1}{2}\mathbf{q}\cdot Df_{\mathbf{p}'}\cdot\mathbf{q}f_{\mathbf{p}}(1-\theta f_{\mathbf{p}})\right.\\ &\left. -\left(\mathbf{q}\cdot\frac{\partial f_{\mathbf{p}}}{\partial\mathbf{p}}\right)\left(\mathbf{q}\cdot\frac{\partial f_{\mathbf{p}'}}{\partial\mathbf{p}'}\right)(1-\theta f_{\mathbf{p}})(1-\theta f_{\mathbf{p}'})\right].\end{aligned} \tag{A2}$$

The term of first order vanishes upon integration over $\mathbf{q}$ and the terms of second order is

$$\begin{aligned}&\left|\frac{v(q/\hbar)}{\epsilon\left(q/\hbar, \dfrac{\mathbf{q}\cdot\mathbf{p}}{\hbar m}+\dfrac{q^2}{2m}\right)}\right|^2 \delta\big(\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}'+\mathbf{q})\big)\left[f_{\mathbf{p}+\mathbf{q}}f_{\mathbf{p}'-\mathbf{q}}(1-\theta f_{\mathbf{p}})(1-\theta f_{\mathbf{p}'}) - f_{\mathbf{p}}f_{\mathbf{p}'}(1-\theta f_{\mathbf{p}+\mathbf{q}})(1-\theta f_{\mathbf{p}'-\mathbf{q}})\right]\\ &\approx \frac{1}{2}\left|\frac{v(q/\hbar)}{\epsilon\left(q/\hbar, \dfrac{\mathbf{q}\cdot\mathbf{p}}{\hbar m}\right)}\right|^2 \left(\frac{\partial}{\partial\mathbf{p}}-\frac{\partial}{\partial\mathbf{p}'}\right)\cdot\left\{\mathbf{q}\,\delta\big(\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')\big)\mathbf{q}\cdot\left[\frac{\partial f_{\mathbf{p}}}{\partial\mathbf{p}}f_{\mathbf{p}'}(1-\theta f_{\mathbf{p}'}) - \frac{\partial f_{\mathbf{p}'}}{\partial\mathbf{p}'}f_{\mathbf{p}}(1-\theta f_{\mathbf{p}})\right]\right\}\\ &+\frac{\mathbf{q}}{2}\cdot\frac{\partial}{\partial\mathbf{p}}\left|\frac{v(q/\hbar)}{\epsilon\left(q/\hbar, \dfrac{\mathbf{q}\cdot\mathbf{p}}{\hbar m}\right)}\right|^2 \times\delta\big(\mathbf{q}\cdot(\mathbf{p}-\mathbf{p}')\big)\left[\mathbf{q}\cdot\frac{\partial f_{\mathbf{p}}}{\partial\mathbf{p}}f_{\mathbf{p}'}(1-\theta f_{\mathbf{p}'}) - \mathbf{q}\cdot\frac{\partial f_{\mathbf{p}'}}{\partial\mathbf{p}'}(1-\theta f_{\mathbf{p}})f_{\mathbf{p}}\right].\end{aligned}$$

Hence, to lowest order in $\mathbf{q}$, we find

$$\mathcal{C}^{qB}[f,f](\mathbf{p}) \approx m\pi\frac{\partial}{\partial\mathbf{p}}\cdot\int d\mathbf{p}'\int\frac{d\mathbf{k}}{(2\pi)^3}\left|\frac{v(k)}{\epsilon\left(k, \dfrac{\mathbf{k}\cdot\mathbf{p}}{m}\right)}\right|^2 \times\delta\big(\mathbf{k}\cdot(\mathbf{p}-\mathbf{p}')\big)\mathbf{kk}\cdot\left[\frac{\partial f_{\mathbf{p}}}{\partial\mathbf{p}}f_{\mathbf{p}'}(1-\theta f_{\mathbf{p}'}) - \frac{\partial f_{\mathbf{p}'}}{\partial\mathbf{p}'}f_{\mathbf{p}}(1-\theta f_{\mathbf{p}})\right]. \tag{A3}$$

Equation (A3) can be regarded as the quantum extension of the classical Lenard-Balescu collision integral (in the literature, Eq. (A1) is often abusively referred to as the quantum Lenard-Balescu equation[15]). Like in the classical case, the qLFP equation is obtained in the static limit $\epsilon(k, \frac{\mathbf{k}\cdot\mathbf{p}}{m}) \to \epsilon(k,0)$, leading to

$$\mathcal{C}^{qLFP}[f,f](\mathbf{p}) = \frac{m}{8\pi^2}\frac{\partial}{\partial\mathbf{p}}\cdot\int d\mathbf{p}'\,\overleftrightarrow{G}(\mathbf{p}-\mathbf{p}')\cdot\left[\frac{\partial f_{\mathbf{p}}}{\partial\mathbf{p}}f_{\mathbf{p}'}(1-\theta f_{\mathbf{p}'}) - \frac{\partial f_{\mathbf{p}'}}{\partial\mathbf{p}'}f_{\mathbf{p}}(1-\theta f_{\mathbf{p}})\right]$$

where

$$\overleftrightarrow{G}(\mathbf{g}) = \int d\mathbf{k}\left|\frac{v(k)}{\epsilon(k,0)}\right|^2\delta(\mathbf{k}\cdot\mathbf{g})\mathbf{kk} = \pi(4\pi q^2)^2\ln\Lambda\,\frac{g^2\overleftrightarrow{\mathrm{I}}-\mathbf{gg}}{g^3},$$

Reuse of AIP Publishing content is subject to the terms at: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 192.12.184.6 On: Tue, 22 Mar 2016 15:05:36

with $\mathbf{g} = \mathbf{p} - \mathbf{p}'$, and

$$\ln\Lambda = \int_0^\infty \frac{dk}{k}\left|\frac{1}{\epsilon(k,0)}\right|^2 \tag{A4}$$

is the Coulomb logarithm.

APPENDIX B: IMPORTANT PROPERTIES OF THE FERMI-DIRAC FUNCTION

The quantum equilibrium distribution function

$$f_q(\mathbf{p};\beta,\mu) = \frac{1}{\theta}\frac{1}{e^{-\beta\left(\mu-\frac{\mathbf{p}^2}{2m}\right)} - \delta}$$

satisfies

$$\begin{aligned}\frac{1}{\beta}\frac{\partial}{\partial\mu}f_q &= f_q(1+\delta\theta f_q)\\ \frac{1}{\beta^2}\frac{\partial^2}{\partial\mu^2}f_q &= f_q(1+\delta\theta f_q)(1+2\delta\theta f_q)\\ \frac{\partial}{\partial\mathbf{p}}f_q &= -\frac{\beta}{m}\mathbf{p}f_q(1+\delta\theta f_q).\end{aligned} \tag{B1}$$

We emphasize these properties because the fact that products of the form $f_q(1+\delta\theta f_q)$ and $f_q(1+\delta\theta f_q)(1+2\delta\theta f_q)$ can be simply expressed in term of derivatives of f is essential in deriving the most of the results of the main text. This property of the equilibrium quantum distribution function is quite fortunate. In the classical case, the equivalent properties satisfied by the Maxwell-Boltzmann distribution (more precisely of the underlying exponential function) are even simpler and are often unnoticed, but they are similarly essential to our ability to write closed-form formulas.

In addition, the majority of the closed-form results were obtained by noticing the following relation between the quantum distribution function and the Maxwell-Boltzmann distribution function

$$\begin{aligned}f_q(\mathbf{p};\beta,\mu) &= \frac{(2\pi m)^{3/2}}{n}\frac{1}{\theta}\int_{-\infty}^{\infty} dE\,\frac{1}{e^{-\beta(\mu-E)}-\delta}\\ &\quad\times\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{e^{zE}}{z^{3/2}}f_{cl}(\mathbf{p};z)\,,\end{aligned} \tag{B2}$$

where

$$f_{cl}(\mathbf{p};\beta) = n\left(\frac{\beta}{2\pi m}\right)^{3/2} e^{-\frac{\beta}{2m}\mathbf{p}^2}.$$

Equation (B2) simply follows from $\frac{1}{1+e^{a-x^2}} = \int_{-\infty}^{+\infty}\frac{1}{e^{a-y}-\delta}\delta(y-x^2)$ and $\delta(y-x^2) = \int_{-\infty}^{\infty}\frac{dt}{2\pi}e^{it(y-x^2)} = \int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}e^{zy}e^{-zx^2} = \int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{e^{zy}}{z^{3/2}}z^{3/2}e^{-zx^2}$. The relation (B2) provides a link between the classical results and the quantum results. Indeed, the classical expression for the Rosenbluth potentials and related quantities is momentum integrals of the form

$$I_{cl}(\beta) = \int d\mathbf{v}\,Q(\mathbf{v})\tilde{f}_{cl}(\mathbf{v};\beta),$$

while their quantum counterparts (10) are of the form

$$I_q(\beta,\mu) = \int d\mathbf{v}\,Q(\mathbf{v})\tilde{f}_q(\mathbf{v};\beta,\mu),$$

and

$$\begin{aligned}J_q(\beta,\mu) &= \int d\mathbf{v}\,Q(\mathbf{v})\tilde{f}_q(\mathbf{v};\beta)\left(1+\delta\tilde{\theta}\tilde{f}_q(\mathbf{v};\beta,\mu)\right)\\ &= \frac{1}{\beta}\frac{\partial}{\partial\mu}I_q(\beta,\mu)\,,\end{aligned} \tag{B3}$$

where we used Eq. (B1) in the last expression. Knowing $I_{cl}(\beta)$, I_q, and J_q can be obtained using

$$\begin{aligned}I_q(\beta,\mu) &= \frac{(2\pi m)^{3/2}}{n}\frac{1}{\theta}\int_{-\infty}^{\infty} dE\,\frac{1}{e^{-\beta(\mu-E)}-\delta}\\ &\quad\times\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{e^{zE}}{z^{3/2}}I_{cl}(z)\,,\end{aligned} \tag{B4}$$

which results from the relation (B2). This way the quantum calculation amounts first to an integral in the complex planes, which, for the cases of interest here, can be done using Cauchy's residues theorem. The remaining integral over E yields to Fermi integrals.

APPENDIX C: CALCULATION OF POTENTIALS *H*, *I*, AND *G* IN LOCAL THERMAL EQUILIBRIUM

All closed-form expressions for the potentials H, I, and G are obtained by applying the method outlined in Appendix B.

1. Potential *H*

We have

$$\begin{aligned}H(\mathbf{v}) &= \int d\mathbf{v}'\frac{1}{||\mathbf{v}-\mathbf{v}'||}f_q(\mathbf{v}';\beta)\\ &= \frac{(2\pi m)^{3/2}}{n}\frac{1}{\theta}\int_{-\infty}^{\infty} dE\,\frac{1}{e^{-\beta(\mu-E)}-\delta}\\ &\quad\times\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{e^{zE}}{z^{3/2}}H_{cl}(\mathbf{v};z)\,,\end{aligned}$$

where

$$H_{cl}(\mathbf{v},\beta) = \int d\mathbf{v}'\frac{1}{||\mathbf{v}-\mathbf{v}'||}\tilde{f}_{cl}(\mathbf{v}';\beta) = \frac{n}{v}\mathrm{erf}\left(\sqrt{\frac{m\beta}{2}}\mathrm{v}\right)$$

is the classical Rosenbluth potential. The complex integral is performed as follows:

$$\begin{aligned}\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{e^{zE}}{z^{3/2}}H_{cl}(\mathbf{v};z) &= \frac{\sqrt{2mn}}{\sqrt{\pi}v}\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{e^{zE}}{z}\int_0^v e^{-\frac{mz}{2}x^2}dx\\ &= \frac{\sqrt{2mn}}{\sqrt{\pi}v}\int_0^v dx\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{e^{z\left(E-mx^2/2\right)}}{z}\\ &= \frac{\sqrt{2mn}}{\sqrt{\pi}v}\int_0^v dx\,\Theta\left(E-mx^2/2\right),\end{aligned}$$

Reuse of AIP Publishing content is subject to the terms at: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 192.12.184.6 On: Tue, 22 Mar 2016 15:05:36

where we used

$$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{dz}{2\pi i}\frac{e^{zx}}{z}=\Theta(x). \tag{C1}$$

Therefore,

$$\begin{aligned}H(\mathbf{v})&=\frac{4\pi m}{v}\frac{1}{\theta}\int_0^v dx\int_{mx^2/2}^{\infty}dE\frac{1}{e^{-\beta(\mu-E)}-\delta}\\&=\frac{4\pi m^2}{\beta\theta}\mathcal{Q}_0^c\left(\beta\mu,\frac{m\beta v^2}{2}\right)\\&\quad+\frac{1}{v\theta}\left(\frac{2\pi m}{\beta}\right)^{3/2}\mathcal{Q}_{1/2}\left(\beta\mu,\frac{m\beta v^2}{2}\right),\end{aligned} \tag{C2}$$

after an integration by parts.

2. Potential *I*

The potential I is obtained by applying Eq. (B3) to (C2), i.e.,

$$I(\mathbf{v})=\int d\mathbf{v}'\frac{1}{||\mathbf{v}-\mathbf{v}'||}\tilde{f}_q(\mathbf{v}';\beta)\left(1+\delta\tilde{\theta}\tilde{f}_q(\mathbf{v}';\beta)\right)=\frac{1}{\beta}\frac{\partial}{\partial\mu}H(\mathbf{v}).$$

3. Potential *G*

We again apply Eqs. (B3) and (B4), i.e.,

$$G(\mathbf{v})=\int d\mathbf{v}|\mathbf{v}-\mathbf{v}'|\tilde{f}_q(\mathbf{v}')\left[1+\delta\tilde{\theta}\tilde{f}_q(\mathbf{v})\right]=\frac{1}{\beta}\frac{\partial}{\partial\mu}g(\mathbf{v}),$$

where

$$g(\mathbf{v})=\int d\mathbf{v}'||\mathbf{v}-\mathbf{v}'||\tilde{f}_q(\mathbf{v}';\beta) \tag{C3}$$

$$=\frac{(2\pi m)^{3/2}}{n}\frac{1}{\theta}\int_{-\infty}^{\infty}dE\frac{1}{e^{-\beta(\mu-E)}-\delta}\times\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{e^{zE}}{z^{3/2}}g_{cl}(\mathbf{v};z), \tag{C4}$$

and g_{cl} is the classical Rosenbluth potential

$$\begin{aligned}g_{cl}(\mathbf{v},\beta)&=\int d\mathbf{v}'||\mathbf{v}-\mathbf{v}'||\tilde{f}_{cl}(\mathbf{v}';\beta)\\&=n\left[\left(v+\frac{1}{m\beta v}\right)\mathrm{erf}\left(\sqrt{\frac{m\beta}{2}}\mathrm{v}\right)+\sqrt{\frac{2}{\pi m\beta}}e^{-\frac{m\beta}{2}\mathrm{v}^2}\right].\end{aligned}$$

The complex integral is

$$\begin{aligned}&\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{e^{zE}}{z^{3/2}}g_{cl}(\mathbf{v};z)\\&=n\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{e^{zE}}{z^{3/2}}\times\left[\left(v+\frac{1}{mvz}\right)\sqrt{\frac{2mz}{\pi}}\int_0^v e^{-\frac{m\beta}{2}x^2}dx+\sqrt{\frac{2}{\pi mz}}e^{-\frac{mz}{2}v^2}\right]\\&=\sqrt{\frac{2m}{\pi}}nv\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{e^{zE}}{z}\int_0^v e^{-\frac{m\beta}{2}x^2}dx+\sqrt{\frac{2}{m\pi}}\frac{n}{v}\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{e^{zE}}{z^2}\\&\quad\int_0^v e^{-\frac{m\beta}{2}x^2}dx+\sqrt{\frac{2}{\pi m}}n\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{e^{zE}}{z^2}e^{-z\frac{mv^2}{2}}.\end{aligned}$$

Using $\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{dz}{2\pi i}\frac{e^{zx}}{z^2}=-x\Theta(x)$ and Eq. (C1), we then find

$$\begin{aligned}g(\mathbf{v})&=v^2H(\mathbf{v})+\frac{4\pi}{v}\frac{1}{\theta}\int_0^v dx\int_0^{\infty}\frac{E}{1+e^{-\beta\left(\mu-\frac{mx^2}{2}-E\right)}}\\&\quad+4\pi m\frac{1}{\theta}\int_0^{\infty}\frac{E}{1+e^{-\beta\left(\mu-\frac{mp^2}{2}-E\right)}}.\end{aligned}$$

Using $\int_0^{\infty}dE\frac{1}{1+e^{-\beta(\mu-\frac{mx^2}{2}-E)}}=\ln\left(1+e^{\beta(\mu-\frac{mx^2}{2})}\right)$ and Eq. (B1), we find

$$\begin{aligned}G(\mathbf{v})&=v^2I(\mathbf{v})+\frac{1}{\beta}\left(\frac{2\pi m}{\beta}\right)^{3/2}\frac{1}{v}\mathcal{Q}_{1/2}\left(\beta\mu,\frac{m\beta}{2}v^2\right)\\&\quad+\frac{8\pi m}{\beta^2}\frac{1}{\theta}\ln\left(1+e^{\beta\left(\mu-\frac{mv^2}{2}\right)}\right),\end{aligned}$$

where we used an integration by parts.

APPENDIX D: FORMULAS FOR R_{ei} AND Q_{ei}

1. Friction force R_{ei}

We again apply the method outlined in Appendix B to

$$\begin{aligned}\mathbf{F}_{ei}&=\int d\mathbf{p}\mathbf{p}C_{ei}[f_q](\mathbf{p})\\&=\int d\mathbf{p}\mathbf{A}_{ei}(\mathbf{p})f_q(\mathbf{p})\left[\left(1+\frac{m_i}{m_e}\right)-\theta_e f_q(\mathbf{p})\right]\\&=\frac{m_i}{m_e}\mathbf{f}_{ei}+\frac{1}{\beta_e}\frac{\partial}{\partial\mu_e}\mathbf{f}_{ei},\end{aligned} \tag{D1}$$

where

$$\begin{aligned}\mathbf{f}_{ei}&\equiv\int d\mathbf{p}\mathbf{A}_{ei}(\mathbf{p})f_q(\mathbf{p})\\&=\frac{(2\pi m_e)^{3/2}}{n_e}\frac{1}{\theta_e}\int_{-\infty}^{\infty}dE\frac{1}{1+e^{-\beta_e(\mu_e-E)}}\\&\quad\times\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{e^{zE}}{z^{3/2}}\mathbf{f}_{ei}^{cl}(z)\end{aligned} \tag{D2}$$

and

$$\mathbf{f}_{ei}^{cl}\equiv\int d\mathbf{p}\mathbf{A}_{ei}(\mathbf{p})f_{cl}(\mathbf{p}).$$

In the following, we first determine $\mathbf{f}_{ei}^{cl}$ and substitute the result into Eqs. (D2) and (D1).

a. Evaluation of $\mathbf{f}_{ei}^{cl}$

We will first show that

$$\mathbf{f}_{ei}^{cl}=\frac{\gamma^{ei}n_en_i}{m_i\mu_{ei}}\frac{\partial}{\partial\mathbf{u}}\left[\frac{1}{u}\mathrm{erf}\left(\frac{u}{\left(\mathrm{v}_i^2+\mathrm{v}_e^2\right)^{1/2}}\right)\right] \tag{D3}$$

$$\approx-\frac{\gamma^{ei}}{m_i\mu_{ei}}\frac{4n_in_e}{3\sqrt{\pi}}\frac{1}{\left(v_e^2+v_i^2\right)^{3/2}}(\mathbf{u}_e-\mathbf{u}_i) \tag{D4}$$

$$\text{if } ||\mathbf{u}_i-\mathbf{u}_e||/v_i\ll 1,$$

Reuse of AIP Publishing content is subject to the terms at: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 192.12.184.6 On: Tue, 22 Mar 2016 15:05:36

where $\mathbf{u} = \mathbf{u}_e - \mathbf{u}_e$. In the literature, one generally finds the approximate expression (D4) for $\mathbf{f}^{cl}_{ei}$. For our purpose, we use the exact expression (D3) since Eq. (D2) requires an integral of $\mathbf{f}^{cl}_{ei}(z)$ over the entire range of inverse temperature z, and the approximation (D4) is not valid across the entire range.

Proof. Using $\mathbf{A}_{ei}$ and $H_{ei} = \frac{n_i}{\|\mathbf{v}-\mathbf{u}_i\|}\mathrm{erf}\left(\sqrt{\frac{\beta_i m_i}{2}}\|\mathbf{v}-\mathbf{u}_e)\|\right)$, after change of variables and defining $\mathbf{u} = \mathbf{u}_e - \mathbf{u}_e$

$$\begin{aligned}\mathbf{f}^{cl}_{ei} &= \left(\frac{m_e\beta_e}{2\pi}\right)^{3/2}\frac{\gamma^{ei}n_e n_i}{m_i\mu_{ei}}\frac{\partial}{\partial\mathbf{u}}\int d\mathbf{v}\frac{\mathrm{erf}(\mathrm{v}/\mathrm{v}_i)}{v}e^{-(\mathbf{v}-\mathbf{u})^2/v_e^2}\\ &= \left(\frac{m_e\beta_e}{2\pi}\right)^{3/2}\frac{\gamma^{ei}n_e n_i}{m_i\mu_{ei}}\\ &\quad\times\frac{\partial}{\partial\mathbf{u}}\left[\frac{\pi v_e^2}{u}\int_{-\infty}^{\infty}d\mathrm{v}\,\mathrm{erf}(\mathrm{v}/\mathrm{v}_i)e^{-(\mathrm{v}-\mathrm{v})^2/\mathrm{v}_e^2}\right]\\ &= \left(\frac{m_e\beta_e}{2\pi}\right)^{3/2}\frac{\gamma^{ei}n_e n_i}{m_i\mu_{ei}}\frac{\partial}{\partial\mathbf{u}}\left[\frac{\pi^{3/2}v_e^3}{u}\mathrm{erf}\left(\frac{1}{(\mathrm{v}_e^2+\mathrm{v}_i^2)^{1/2}}u\right)\right].\end{aligned}$$

□

b. Evaluation of $\mathbf{f}_{ei}$

Using Eq. (D3) into Eq. (D2), we find

$$\begin{aligned}\mathbf{f}_{ei} &= \frac{(2\pi m_e)^{3/2}}{n_e}\frac{\gamma^{ei}n_e n_i}{m_i\mu_{ei}}\sqrt{\frac{m_e}{2}}\frac{1}{\theta_e}\int_{-\infty}^{\infty}dE\frac{1}{1+e^{-\beta_e(\mu_e-E)}}\\ &\quad\times\frac{\partial}{\partial\mathbf{u}}\left[\frac{1}{u}\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{e^{\frac{2E}{m}z}}{z^{3/2}}\mathrm{erf}\left(\left(\frac{\mathrm{z}}{1+\mathrm{z}\mathrm{v}_i^2}\right)^{1/2}u\right)\right].\end{aligned}$$

This integral can be simplified in the limit $\|\mathbf{u}_i - \mathbf{u}_e\|/v_i \ll 1$. Indeed, for all z, $\left(\frac{z}{1+zv_i^2}\right)^{1/2}u \le \|\mathbf{u}_i - \mathbf{u}_e\|/v_i$. When $\|\mathbf{u}_i - \mathbf{u}_e\|/v_i \ll 1$,

$$\begin{aligned}\mathbf{f}_{ei} &= \frac{(2\pi m_e)^{3/2}}{n_e}\frac{\gamma^{ei}n_e n_i}{m_i\mu_{ei}}\sqrt{\frac{m_e}{2}}\frac{1}{\theta_e}\int_{-\infty}^{\infty}dE\frac{1}{1+e^{-\beta_e(\mu_e-E)}}\\ &\quad\times -\frac{4}{3\sqrt{\pi}}\mathbf{u}\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{e^{\frac{2E}{m}z}}{(1+v_i^2 z)^{3/2}}.\end{aligned}$$

The complex integral is calculated in Appendix G 1, which yields

$$\begin{aligned}\mathbf{f}_{ei} &= -\frac{1}{n_e}\frac{\gamma^{ei}n_e n_i}{m_i\mu_{ei}}\sqrt{\frac{m_e}{2}}\frac{8}{3\pi}\left(\frac{\pi m_e m_i\beta_i}{2}\right)^{3/2}\\ &\quad\times\sqrt{\frac{m_e}{2}}\left(\frac{m_e}{m_i\beta_i}\right)^{3/2}\frac{\mathbf{u}}{\theta_e}\int_0^{\infty}dx\frac{\sqrt{x}e^{-x}}{1+e^{-\beta_e\mu_e}e^{\frac{m_e\beta_e}{m_i\beta_i}x}}.\end{aligned}$$

c. Evaluation of $\mathbf{F}_{ei}$

$\mathbf{F}_{ei}$ results form applying Eq. (D1) to $\mathbf{f}_{ei}$ above.

2. Collisional energy exchange rate Q_{ei}

Following the method outlined in Appendix B, we write

$$\begin{aligned}Q_{ei} &= \int d\mathbf{p}\frac{\mathbf{p}^2}{2m_e}C_{ei}(\mathbf{p}) - \mathbf{u}_{ei}\cdot\mathbf{F}_{ei}\\ &= \int d\mathbf{p}\left\{\frac{\mathbf{p}}{m_e}\cdot\mathbf{A}_{ei}(\mathbf{p})f_q(\mathbf{p})\left[\left(1+\frac{m_i}{m_e}\right)-\theta_e f_q(\mathbf{p})\right]\right.\\ &\quad\left.+\frac{1}{2m_e}\mathrm{Tr}\overleftrightarrow{D}_{ei}(\mathbf{p})f_q(\mathbf{p})\right\} - \mathbf{u}_{ei}\cdot\mathbf{F}_{ei}\\ &\equiv \frac{m_i}{m_e}q_1 + \frac{1}{\beta_e}\frac{\partial}{\partial\mu_e}q_1 + q_2 - \mathbf{u}_{ei}\cdot\mathbf{F}_{ei},\end{aligned}$$

where

$$\begin{aligned}q_1 &= \frac{1}{m_e}\int d\mathbf{p}\,\mathbf{p}\cdot\mathbf{A}_{ei}(\mathbf{p})f_q(\mathbf{p})\\ &= \frac{(2\pi m_e)^{3/2}}{n_e}\frac{1}{\theta_e}\int_{-\infty}^{\infty}dE\frac{1}{1+e^{-\beta_e(\mu_e-E)}}\\ &\quad\times\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{e^{zE}}{z^{3/2}}q_1^{cl}(z)\\ q_2 &= \frac{1}{2m_e}\int d\mathbf{p}\,\mathrm{Tr}\overleftrightarrow{D}_{ei}(\mathbf{p})f_q(\mathbf{p})\\ &= \frac{(2\pi m_e)^{3/2}}{n_e}\frac{1}{\theta_e}\int_{-\infty}^{\infty}dE\frac{1}{1+e^{-\beta_e(\mu_e-E)}}\\ &\quad\times\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{e^{zE}}{z^{3/2}}q_2^{cl}(z)\end{aligned}$$

and

$$\begin{aligned}q_1^{cl}(\beta_e) &= \frac{1}{m_e}\int d\mathbf{p}\,\mathbf{p}\cdot\mathbf{A}_{ei}(\mathbf{p})f_B(\mathbf{p})\\ q_2^{cl}(\beta_e) &= \frac{1}{2m_e}\int d\mathbf{p}\mathrm{Tr}\overleftrightarrow{D}_{ei}(\mathbf{p})f_B(\mathbf{p}).\end{aligned}$$

a. Evaluation of q_1^{cl} *and* q_2^{cl}

Following calculations similar to those previously outlines for the calculation of $\mathbf{r}^{cl}_{ei}$, we find

$$\begin{aligned}q_1^{cl}(\beta_e) &= \mathbf{u}_i\cdot\mathbf{r}^{cl}_{ei} + \frac{n_e n_i\gamma^{ei}}{m_i\mu_{ei}}\times\left[-\frac{\mathrm{erf}\left(\frac{u}{(\mathrm{v}_e^2+\mathrm{v}_i^2)^{1/2}}\right)}{u}\right.\\ &\quad\left.+\frac{2}{\sqrt{\pi}}\frac{1}{(v_e^2+v_i^2)^{1/2}}\frac{1}{1+v_i^2/v_e^2}e^{-\frac{u^2}{v_e^2+v_i^2}}\right]\\ q_2^{cl}(\beta_e) &= \frac{n_e n_i\gamma^{ei}}{m_e\mu_{ei}}\frac{\mathrm{erf}\left(\frac{u}{(\mathrm{v}_e^2+\mathrm{v}_i^2)^{1/2}}\right)}{u}.\end{aligned}$$

In the limit $\|\mathbf{u}_i - \mathbf{u}_e\|/v_i \ll 1$

$$q_1^{cl}(\beta_e) = -\frac{2}{\sqrt{\pi}}\sqrt{\frac{m_e}{2}}\frac{\gamma^{ei}n_e n_i}{m_i\mu_{ei}}\frac{\beta_e^{1/2}}{\left(1+\frac{m_e\beta_e}{m_i\beta_i}\right)^{3/2}},$$

$$q_2^{cl}(\beta_e) = \frac{2m_i}{m_e\sqrt{\pi}}\sqrt{\frac{m_e}{2}}\frac{\gamma^{ei}n_e n_i}{m_i\mu_{ei}}\frac{\beta_e^{1/2}}{\left(1+\frac{m_e\beta_e}{m_i\beta_i}\right)^{1/2}}.$$

Reuse of AIP Publishing content is subject to the terms at: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 192.12.184.6 On: Tue, 22 Mar 2016 15:05:36

b. Evaluation of $\mathbf{q}_1$ *and* $\mathbf{q}_2$

$$q_1 = -\frac{(2\pi m_e)^{3/2}}{n_e}\sqrt{\frac{m_e}{2}}\frac{2}{\sqrt{\pi}}\frac{\gamma^{ei}n_e n_i}{m_i\mu_{ei}}\frac{1}{\theta_e} \times \int_{-\infty}^{\infty} dE \frac{1}{1+e^{-\beta_e(\mu_e-E)}}\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{e^{\frac{2E}{m_e}z}}{z\left(1+v_i^2 z\right)^{3/2}},$$

$$q_2 = \frac{(2\pi m_e)^{3/2}}{n_e}\sqrt{\frac{m_e}{2}}\frac{2}{\sqrt{\pi}}\frac{\gamma^{ei}n_e n_i}{m_i\mu_{ei}}\frac{m_i}{m_e}\frac{1}{\theta_e} \times \int_{-\infty}^{\infty} dE \frac{1}{1+e^{-\beta_e(\mu_e-E)}}\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{e^{\frac{2E}{m_e}z}}{z\left(1+v_i^2 z\right)^{1/2}}.$$

The complex integrals are calculated in Appendix G 2.

c. Evaluation of $\mathbf{Q}_{ei}$

In the limit $||\mathbf{u}_i - \mathbf{u}_e||/v_i \ll 1$

$$Q_{ei} = 16\sqrt{\pi}\frac{\gamma^{ei}}{\mu_{ei}}n_i\frac{m_e^2}{m_i}(k_B T_e - k_B T_i) \times \frac{1}{\theta_e}\int_0^{\infty} dx \frac{x^2 e^{-x^2}}{1+e^{-\beta_e\mu_e}e^{\frac{m_e\beta_e}{m_i\beta_i}x^2}}.$$

In the additional limit $m_e/m_i \ll 1$

$$Q_{ei} = 4\pi\frac{\gamma^{ei}n_i m_e}{m_i}\frac{1}{\theta_e}\frac{1}{1+e^{-\beta_e\mu_e}}(k_B T_e - k_B T_i).$$

APPENDIX E: SELF-ADJOINTNESS

We use the notations introduced in Sec. III

1. $\mathcal{C}_1$, $\mathcal{C}_2$

The proof is straightforward once we observe the following relations between the friction and diffusion terms

$$\mathbf{C}f_0(1+\delta\theta f_0)a = \frac{1}{\beta}\frac{\partial}{\partial\mu}\left\{\mathbf{A}_{ee}[f_0]f_0(1+\delta\theta f_0)a\right\} \quad \text{(E1a)}$$

and

$$\frac{\partial}{\partial\mathbf{p}}\cdot\overleftrightarrow{D}[f_0] = 2\mathbf{B}_{ee}[f_0] = \frac{2}{\beta}\frac{\partial}{\partial\mu}\mathbf{A}_{ee}[f_0], \quad \text{(E1b)}$$

$$\overleftrightarrow{D}[f_0]\cdot\mathbf{p} = -\frac{2m}{\beta}\mathbf{A}_{ee}[f_0], \quad \text{(E1c)}$$

$$\frac{2}{\beta}\frac{\partial}{\partial\mu}\mathbf{A}_{ee}[af_0] = \frac{\partial}{\partial\mathbf{p}}\cdot\overleftrightarrow{D}[af_0(1+\delta\theta f_0)], \quad \text{(E1d)}$$

$$\frac{1}{\beta^2}\frac{\partial^2}{\partial\mu^2}\mathbf{A}_{ee}[af_0] = \mathbf{A}_{ee}[af_0(1+\delta\theta f_0)(1+2\delta\theta f_0)], \quad \text{(E1e)}$$

$$\frac{1}{\beta^2}\frac{\partial^2}{\partial\mu^2}\overleftrightarrow{D}[af_0] = \overleftrightarrow{D}[af_0(1+\delta\theta f_0)(1+2\delta\theta f_0)], \quad \text{(E1f)}$$

which are direct consequences of the basic properties (B1) satisfied by the quantum distribution function f_0.

The relations (E1a)–(E1c) imply

$$\mathcal{C}_1[f_0, af_0(1+\delta\theta f_0)] = \frac{1}{2}\frac{\partial}{\partial\mathbf{p}}\cdot\left[\overleftrightarrow{D}[f_0]\cdot\frac{\partial a}{\partial\mathbf{p}}f_0(1+\delta\theta f_0)\right], \quad \text{(E2)}$$

while (E1d)–(E1f) yield

$$\mathcal{C}_2[af_0(1+\delta\theta f_0), f_0] = \gamma^{ee}\frac{\partial}{\partial\mathbf{p}}\cdot\left\{\left[\int d\mathbf{p}'\overleftrightarrow{V}(\mathbf{p},\mathbf{p}')\cdot\frac{\partial a'}{\partial\mathbf{p}'}f_0'\left(1+\delta\theta f_0'\right)\right]f_0(1+\delta\theta f_0)\right\}. \quad \text{(E3)}$$

By integration by parts, Equations (E2) and (E3) give the desired relations

$$[b,a]_1 = -\gamma^{ee}\int\int d\mathbf{p}d\mathbf{p}'\overleftrightarrow{V}(\mathbf{p},\mathbf{p}'):\left[\frac{\partial b}{\partial\mathbf{p}}\frac{\partial a}{\partial\mathbf{p}}\right] \times f_0'(1+\delta\theta f_0')f_0(1+\delta\theta f_0) = [a,b]_1$$

and

$$[b,a]_2 = -\gamma^{ee}\int\int d\mathbf{p}d\mathbf{p}'\overleftrightarrow{V}(\mathbf{p},\mathbf{p}'):\left[\frac{\partial b}{\partial\mathbf{p}}\frac{\partial a'}{\partial\mathbf{p}'}\right] \times f_0'(1+\delta\theta f_0')f_0(1+\delta\theta f_0) = [a,b]_2,$$

where a and b are any phase-space functions.

2. $f_0(1-\theta f_0)\mathcal{O}[\delta f]$

Defining

$$\mathbf{P}[\delta f] = -\int d\mathbf{p}\, c\mathbf{g}\delta f,$$

$$\mathcal{B}[\delta f] = -\int d\mathbf{p}\,[2cg^2 + d_\parallel + 2d_\perp]\delta f,$$

the approximation defined in Sec. III C 2 becomes

$$\mathcal{O}[\delta f] = \mathbf{K}\cdot\mathbf{P}[\delta f] + \mathcal{E}\mathcal{B}[\delta f].$$

Then,

$$\begin{aligned}[a,b]_{\tilde{2}} &= \int d\mathbf{p}f_0(1-\theta f_0)b\mathbf{K}\cdot\mathbf{P}[f_0(1-\theta f_0)a] + \int d\mathbf{p}f_0(1-\theta f_0)b\mathcal{E}\mathcal{B}[f_0(1-\theta f_0)a] \\ &= \alpha_K\mathbf{P}[f_0(1-\theta f_0)a]\cdot\int d\mathbf{p}f_0(1-\theta f_0)bc\mathbf{g} + \alpha_{\mathcal{E}}\mathcal{B}[f_0(1-\theta f_0)a] \times \int d\mathbf{p}f_0(1-\theta f_0)b[2cg^2+d_\parallel+2d_\perp] \\ &= -\alpha_K\mathbf{P}[f_0(1-\theta f_0)a]\cdot\mathbf{P}[f_0(1-\theta f_0)b] - \alpha_{\mathcal{E}}\mathcal{B}[f_0(1-\theta f_0)a]\mathcal{B}[f_0(1-\theta f_0)b] \\ &= [b,a]_{\tilde{2}}.\end{aligned}$$

Reuse of AIP Publishing content is subject to the terms at: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 192.12.184.6 On: Tue, 22 Mar 2016 15:05:36

APPENDIX F: FORMULAS FOR α_K AND $\alpha_{\mathcal{E}}$

We use the notations introduced in Sec. III.

1. $\alpha_K = \left[\frac{1}{3}\int d\mathbf{p}\, cg^2 f_0(1-\theta f_0)\right]^{-1}$

From Eqs. (B1) and (E1a), with $F_0(g) = f_0(\mathbf{g} + m\mathbf{u})$

$$F_0(1-\theta F_0) = -\frac{m}{\beta g}\frac{\partial}{\partial g}F_0(g), \tag{F1}$$

$$cf_0(1-\theta f_0) = \frac{1}{\beta}\frac{\partial}{\partial \mu}\left[a(g)F_0(g)\left(1-\theta F_0(g)\right)\right],$$

and

$$a(g) = -\frac{4\gamma^{ee}}{g^3}\left(\frac{m}{2\pi\hbar^2\beta}\right)^{\frac{3}{2}}\mathcal{Q}_{\frac{1}{2}}\left(\beta\mu, \frac{\beta}{2m}g^2\right)$$

$$\frac{1}{\alpha_K} = \frac{16\pi\gamma^{ee}m}{3\beta^2}\left(\frac{m}{2\pi\hbar^2\beta}\right)^{\frac{3}{2}} \times \frac{\partial}{\partial\mu}\left[\int_0^\infty dg\,\mathcal{Q}_{\frac{1}{2}}\left(\beta\mu, \frac{\beta}{2m}g^2\right)\frac{\partial F_0}{\partial g}\right]$$
$$= -(2\pi)^{3/2}\frac{16\pi\gamma^{ee}m}{3\beta^2}\int_0^\infty dg g^2 F_0(g)^2$$

after an integration by parts. The last integral is calculated using Eq. (F1) in the form $g^2F_0^2 = \frac{1}{\theta}\left[g^2F_0 + \frac{mg}{\beta}\frac{\partial F_0}{\partial g}\right]$ and an integration by parts, which directly leads to the expression (27) of the main text.

2. $\alpha_{\mathcal{E}} = [\int d\mathbf{p}[2cg^4 + (d_\parallel + 2d_\perp)g^2] f_0(1-\theta f_0)]^{-1}$

Using the expression (21) for a, b, $d_\parallel$, and $d_\perp$, the integrand becomes

$$\frac{1}{\alpha_{\mathcal{E}}} = 4\pi\int_0^\infty dg\left[2(1-2\theta F_0)ag^6 + 16\pi\gamma^{ee}\frac{m}{\beta}g^2F_0\right]F_0(1-\theta F_0).$$

The first part of the integral (which includes a) can be reexpressed with an integration by parts using Eq. (B1), i.e., $\frac{\partial F_0(1-\theta F_0)}{\partial g} = -\frac{\beta}{m}gF_0(1-\theta F_0)$. The resulting integrant involves the term $\frac{\partial}{\partial g}g^5 a(g)$, which can evaluated using Eq. (21). This results in the following expression:

$$\frac{1}{\alpha_{\mathcal{E}}} = -128\pi^2\gamma^{ee}\left(\frac{m}{\beta}\right)^2\int_0^\infty dg g^2 F_0(g)^2,$$

where the last integral was explained in Appendix F 1.

APPENDIX G: EVALUATION OF COMPLEX INTEGRALS

1. $\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{dz}{2\pi i}\frac{e^{tz}}{(a+z)^{3/2}}$

We show that

$$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{dz}{2\pi i}\frac{e^{tz}}{(a+z)^{3/2}} = \frac{2\sqrt{t}}{\sqrt{\pi}}e^{-at}\Theta(t), \tag{G1}$$

where $\alpha \geq 0$, $a > 0$, and $t \in \mathbb{R}$ are real constant.

Proof. The integral can be obtained from the relation

$$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{dz}{2\pi i}\frac{e^{tz}}{(a+z)^{3/2}} = -2\frac{\partial}{\partial a}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{dz}{2\pi i}\frac{e^{tz}}{(a+z)^{1/2}},$$

where, as we will show below, the second integral is

$$\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{dz}{2\pi i}\frac{e^{tz}}{(a+z)^{1/2}} = \frac{1}{\sqrt{\pi t}}e^{-at}\Theta(t),$$

where θ is the Heaviside step function.

The last integral can be evaluated as follows.

(1) *Let us assume* $t \geq 0$. The integrand has a branch point at $z = a$ and we choose the real segment $]-\infty; a]$ as the branch cut. Let us consider

$$\int_{\mathcal{C}}\frac{dz}{2\pi i}\frac{e^{tz}}{(a+z)^{1/2}},$$

where $\mathcal{C}$ is the contour shown in Fig. 6. On the figure, the sections BC and DE actually lie on the real axis but are shown separated for visual purposes. FG is a circle of radius ϵ and centered at $(a, 0)$. The sections AB and EF are arcs of a circle of radius R and centered at the origin. Finally, the thick line to left of $x = a$ on the real axis represents the chosen branch line. Since the integrand is analytic inside and on $\mathcal{C}$, we have by Cauchy's theorem

$$\int_{\mathcal{C}}\frac{dz}{2\pi i}\frac{e^{tz}}{(a+z)^{1/2}} = 0,$$

or, more explicitly in terms of the line integrals along the sections of contour starting from point F

$$\int_{\alpha-iR}^{\alpha+iR}\frac{e^{izt}}{(a+z)^{1/2}}dz + \int_{\phi}^{\pi}\frac{e^{Re^{i\theta}t}}{(a+Re^{i\theta})^{1/2}}iRe^{i\theta}d\theta$$
$$+\int_{R-a}^{\epsilon} -\frac{e^{-(u+a)t}}{i\sqrt{u}}du + \int_{\pi}^{-\pi}\frac{e^{(\epsilon e^{i\theta}-a)t}}{(\epsilon e^{i\theta})^{1/2}}i\epsilon e^{i\theta}d\theta$$
$$+\int_{\epsilon}^{R-a} -\frac{e^{-(u+a)t}}{-i\sqrt{u}}du + \int_{\pi}^{2\pi-\phi}\frac{e^{Re^{i\theta}t}}{(a+Re^{i\theta})^{1/2}}iRe^{i\theta}d\theta = 0.$$

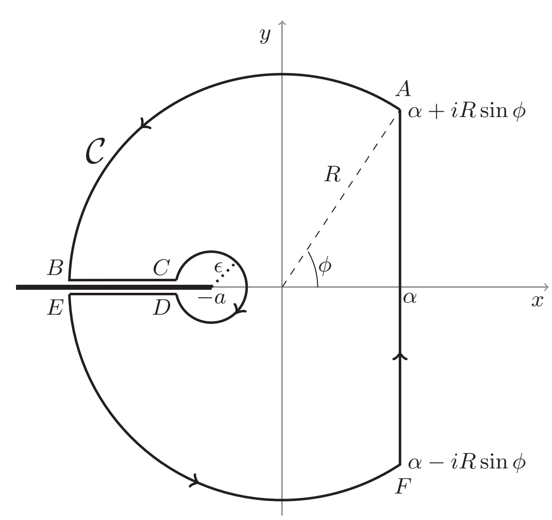

FIG. 6. Contour $\mathcal{C} = ABCDEFA$ used to evaluate integrals G1 and G2.

Reuse of AIP Publishing content is subject to the terms at: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 192.12.184.6 On: Tue, 22 Mar 2016 15:05:36

In the limit $R \to \infty$ and $\epsilon \to 0$, the second, fourth, and sixth contributions (corresponding to integrals along the arcs of circle AB and EF, and along the circle CD) vanish, and we have

$$\begin{aligned}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{dz}{2\pi i} \frac{e^{izt}}{(a+z)^{1/2}} &= \lim_{R\to\infty,\epsilon\to 0} \frac{1}{\pi} \int_{\epsilon}^{R-a} \frac{e^{-(u+a)t}}{u^{1/2}} du \\ &= \frac{2e^{-at}}{\pi} \int_0^{\infty} e^{-v^2 t} dv \quad (u = v^2) \\ &= \frac{e^{-at}}{\sqrt{\pi t}}.\end{aligned}$$

(2) *Let us assume $t < 0$*. Let us consider

$$\int_{\mathcal{C}'} \frac{dz}{2\pi i} \frac{e^{tz}}{(a+z)^{1/2}},$$

where $\mathcal{C}'$ is the contour shown in Fig. 7. Since the integrand is analytic inside and on $\mathcal{C}'$, this integral vanishes by Cauchy's theorem. Moreover, in the limit $R \to \infty$, the integral along the arc of circle FGA vanishes, and therefore

$$\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{dz}{2\pi i} \frac{e^{izt}}{(a+z)^{1/2}} = 0.$$

2. $\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{dz}{2\pi i} \frac{e^{tz}}{z(a+z)^{3/2}}$

We show that

$$\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{dz}{2\pi i} \frac{e^{tz}}{z(a+z)^{3/2}} = \left[\frac{\mathrm{erf}\left(\sqrt{at}\right)}{a^{3/2}} - \frac{2}{\sqrt{\pi}} \frac{\sqrt{t}}{a} e^{-at} \right] \Theta(t), \tag{G2}$$

where $\alpha \geq 0$, $a > 0$ and $t \in \mathbb{R}$ are real constant.

Proof. The integral is obtained from the relation

$$\int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \frac{e^{tz}}{z(a+z)^{3/2}} = -2 \frac{\partial}{\partial a} \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \frac{e^{tz}}{z(a+z)^{1/2}},$$

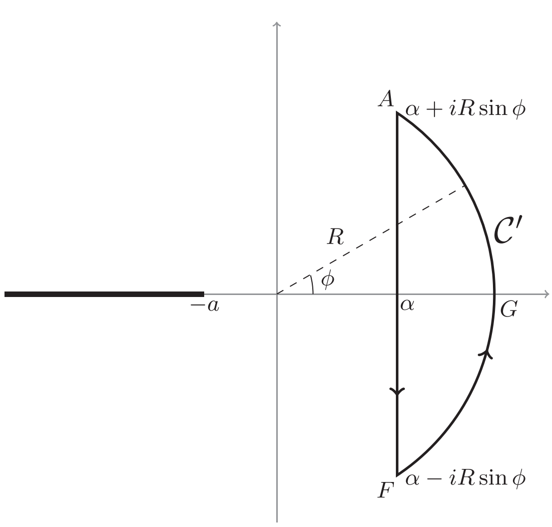

FIG. 7. Contour $\mathcal{C} = AFGA$ used to evaluate integrals G1 and G2.

where, as we show below, the second integral is

$$\int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \frac{e^{tz}}{z(a+z)^{1/2}} = \frac{\mathrm{erf}\left(\sqrt{at}\right)}{\sqrt{a}} \Theta(t).$$

The last integral can be evaluated as follows.

(1) *Let us assume $t \geq 0$*. As before in Appendix G 1, the integrand has a branch point at $z = a$ and we choose the real segment $]-\infty; a]$ as the branch cut. In addition, it has a single pole at the origin. Let us consider

$$\int_{\mathcal{C}} \frac{dz}{2\pi i} \frac{e^{tz}}{(a+z)^{1/2}},$$

where $\mathcal{C}$ is again the contour shown in Fig. 6. By the residue theorem, this contour integral is equal to the residue at the pole $z = 0$

$$\int_{\mathcal{C}} \frac{dz}{2\pi i} \frac{e^{tz}}{z(a+z)^{1/2}} = \mathrm{Res}\left(\frac{e^{tz}}{z(a+z)^{1/2}} \right)_{z=0} = \frac{1}{\sqrt{a}}.$$

As in Appendix G 1, in the limit $R \to \infty$ and $\epsilon \to 0$, the contributions to the contour integral along the arcs of circle AB and EF and along the circle CD vanish. Hence, we are left with

$$\begin{aligned}&\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{dz}{2\pi i} \frac{e^{tz}}{z(a+z)^{1/2}} \\ &= \frac{1}{\sqrt{a}} + \lim_{R\to\infty,\epsilon\to 0} \frac{1}{\pi} \int_{\epsilon}^{R-a} \frac{e^{-(u+a)t}}{(u-a)u^{1/2}} du \\ &= \frac{\mathrm{erf}\left(\sqrt{at}\right)}{\sqrt{a}}.\end{aligned}$$

In evaluating the last integral, we used

$$\begin{aligned}\int_0^{\infty} du \frac{e^{-(u+a)t}}{(u+a)\sqrt{u}} &= \int_t^{\infty} dt' \int_0^{\infty} du \frac{e^{-(u+a)t}}{\sqrt{u}} \\ &= \int_t^{\infty} dt' \frac{\sqrt{\pi} e^{-at'}}{\sqrt{t'}} = \frac{\pi}{\sqrt{a}} \mathrm{erfc}\left(\sqrt{at}\right).\end{aligned}$$

(2) *Let us assume $t < 0$*. As before, the integral

$$\int_{\mathcal{C}'} \frac{dz}{2\pi i} \frac{e^{tz}}{z(a+z)^{1/2}}$$

on the contour $\mathcal{C}'$ shown in Fig. 7 vanishes since the integrand in analytic inside and on the contour. It implies

$$\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{dz}{2\pi i} \frac{e^{izt}}{z(a+z)^{1/2}} = 0.$$

[1]J. Clerouin, G. Robert, P. Arnault, C. Ticknor, J. D. Kress, and L. A. Collins, Phys. Rev. E **91**, 011101 (2015).
[2]R. R. Fäustlin, Th. Bornath, T. Döppner, S. Düsterer, E. Förster, C. Fortmann, S. H. Glenzer, S. Göde, G. Gregori, R. Irsig, T. Laarmann, H. J. Lee, B. Li, K. H. Meiwes-Broer, J. Mithen, B. Nagler, A. Przystawik, H. Redlin, R. Redmer, H. Reinholz, G. Röpke, F. Tavella, R. Thiele, J. Tiggesbäumker, S. Toleikis, I. Uschmann, S. M. Vinko, T. Whitcher, U.

Reuse of AIP Publishing content is subject to the terms at: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 192.12.184.6 On: Tue, 22 Mar 2016 15:05:36

Zastrau, B. Ziaja, and Th. Tschentscher, Phys. Rev. Lett. **104**, 125002 (2010).
[3]Z. Chen, V. Sametoglu, Y. Y. Tsui, T. Ao, and A. Ng, Phys. Rev. Lett. **108**, 165001 (2012).
[4]F. Dorchies, V. Recoules, J. Bouchet, C. Fourment, P. M. Leguay, B. I. Cho, K. Engelhorn, M. Nakatsutsumi, C. Ozkan, T. Tschentscher, M. Harmand, S. Toleikis, M. Störmer, E. Galtier, H. J. Lee, B. Nagler, P. A. Heimann, and J. Gaudin, Phys. Rev. B **92**, 144201 (2015).
[5]A. C. Hayes, G. Jungman, A. E. Schulz, M. Boswell, M. M. Fowler, G. Grim, A. Klein, R. S. Rundberg, J. B. Wilhelmy, D. Wilson, C. Cerjan, D. Schneider, S. M. Sepke, A. Tonchev, and C. Yeamans, Phys. Plasmas **22**, 082703 (2015).
[6]A. B. Zylstra, J. A. Frenje, P. E. Grabowski, C. K. Li, G. W. Collins, P. Fitzsimmons, S. Glenzer, F. Graziani, S. B. Hansen, S. X. Hu, M. Gatu Johnson, P. Keiter, H. Reynolds, J. R. Rygg, F. H. Sguin, and R. D. Petrasso, Phys. Rev. Lett. **114**, 215002 (2015).
[7]J. A. Frenje, P. E. Grabowski, C. K. Li, F. H. Sguin, A. B. Zylstra, M. Gatu Johnson, R. D. Petrasso, V. Yu Glebov, and T. C. Sangster, Phys. Rev. Lett. **115**, 205001 (2015).
[8]J. C. Fernndez, J. J. Honrubia, B. J. Albright, K. A. Flippo, D. C. Gautier, B. M. Hegelich, M. J. Schmitt, M. Temporal, and L. Yin, Nucl. Fusion **49**, 065004 (2009).
[9]*Frontiers and Challenges in Warm Dense Matter*, Lecture Notes in Computational Science and Engineering Vol. 96, edited by F. Graziani, M. P. Desjarlais, R. Redmer, and S. B. Trickey (Springer, New York, 2014).
[10]C. A. Ulrich, *Time-Dependent Density-Functional Theory, Concepts and Applications* (Oxford Graduate Text, 2012).
[11]F. R. Graziani, V. S. Batista, L. X. Benedict, J. I. Castor, H. Chen, S. N. Chen, C. A. Fichtl, J. N. Glosli, P. E. Grabowski, A. T. Graf, S. P. Hau-Riege, A. U. Hazi, S. A. Khairalla, L. Krauss, A. B. Langdon, R. A. London, A. Markmann, M. S. Murillo, D. F. Richards, H. A. Scott, R. Shepherd, L. G. Stanton, F. H. Streitz, M. P. Surh, J. C. Weisheit, and H. D. Whitley, High Energy Density Physics **8**, 105 (2012), and references therein.
[12]H. D. Whitley, Ch. R. Scullard, L. X. Benedict1, J. I. Castor, A. Randles, J. N. Glosli, D. F. Richards, M. P. Desjarlais, and F. R. Graziani, Contrib. Plasma Phys. **55**, 192 (2015).
[13]M. Bonitz, *Quantum Kinetic Theory*, 2nd ed. (Springer, 2016).
[14]D. Kremp, M. Schlanges, and W.-D. Kraeft, *Quantum Statistics of Nonideal Plasmas* (Springer, Heidelberg, 2005).
[15]M. Lampe, Phys. Rev. **170**, 306 (1968).
[16]D. R. Nicholson, *Introduction to Plasma Theory* (John Wiley & Sons, Inc., 1983).
[17]F. L. Hinton, "Collisional transport in plasma," in *Handbook of Plasma Physics*, Basic Plasma Physics Vol. 1, edited by A. A. Galeev and R. N. Sudan (North-Holland Publishing Co., Amsterdam, 1983), p. 148.
[18]M. Lemou, Math. Methods Appl. Sci. **23**, 1093 (2000).
[19]J. Hu, S. Jin, and B. Yan, Commun. Comput. Phys. **12**, 1541 (2012).
[20]J. H. Ferziger and H. G. Kaper, *Mathematical Theory of Transport Processes in Gases* (Elsevier, New York, 1972).
[21]S. Brunner, E. Valeo, and J. A. Krommes, Phys. Plasmas **6**, 4504 (1999).
[22]P. Danielewicz, Physica **100A**, 167 (1980).
[23]E. A. Uehling and G. E. Uhlenbeck, Phys. Rev. **43**, 552 (1933).
[24]S. D. Baalrud and J. Daligault, Phys. Plasmas **21**, 055707 (2014).
[25]S. D. Baalrud and J. Daligault, Phys. Rev. E **91**, 063107 (2015).
[26]J. Daligault, S. D. Baalrud, C. E. Starrett, D. Saumon, and T. Sjostrom, Phys. Rev. Lett. **116**, 075002 (2016).
[27]F. R. Graziani, J. D. Bauer, and M. S. Murillo, Phys. Rev. E **90**, 033104 (2014).
[28]In a dense plasma near equilibrium, an electron with momentum $|\mathbf{p}_1| \approx p_F$, where p_F is the Fermi momentum, can collide only with another electron whose momenta before and after collision, $\mathbf{p}_2$ and $\mathbf{p}_2'$, both lie on the thermally "smeared-out" Fermi surface, i.e., $(1-\Theta)p_F \lesssim |\mathbf{p}_2|, |\mathbf{p}_2|' \lesssim (1+\Theta)p_F$, where $\Theta = k_B T/E_F$ is the degeneracy parameter.
[29]H. W. Wyld and D. Pines, Phys. Rev. **127**, 1851 (1962).
[30]H. Brysk, P. M. Campbell, and P. Hammerling, Plasma Phys. **17**, 473 (1975).
[31]W. B. Hubbard and M. Lampe, Astrophys. J. Suppl. **18**, 297 (1969).
[32]A. Decoster, P. A. Markowich, and B. Perthame, *Modeling of Collisions* (Elsevier, Paris, 1998).
[33]E. E. Salpeter, Aust. J. Phys. **7**, 373 (1954).
[34]This is readily shown performing an integration by parts on the second term in Eq. (3) and writing the first integral as

$$\int d\mathbf{p}_b\, \overleftrightarrow{V}_{ab} \cdot \frac{\partial f_a}{\partial \mathbf{p}_a} f_b[1+\delta_b\theta_b f_b] = \frac{\partial}{\partial \mathbf{p}_a} \cdot \int d\mathbf{p}_b\, \overleftrightarrow{V}_{ab} f_a f_b [1+\delta_b\theta_b f_b] - \int d\mathbf{p}_b\, \frac{\partial}{\partial \mathbf{p}_a} \cdot \overleftrightarrow{V}_{ab} f_a f_b [1+\delta_b\theta_b f_b].$$

[35]M. N. Rosenbluth, W. M. MacDonald, and D. L. Judd, Phys. Rev. **107**, 1 (1957).
[36]M. Goano, ACM Trans. Math. Software **21**, 221 (1995).
[37]We extend the discussion in Ref. 17 to the quantum case.
[38]S. Ichimaru, *Statistical Plasma Physics*, Basic Principles Vol. I (Westview Press, 2004), see Sec. 3–5.
[39]In classical plasma physics textbook,[17] the assumption on $\mathbf{u}_i - \mathbf{u}_e$ is generally expressed in terms of the electronic thermal speed $v_e = \sqrt{2/m_i\beta_i}$, i.e., $|\mathbf{u}_i - \mathbf{u}_e| \ll v_e$. In many applications, $|\mathbf{u}_i - \mathbf{u}_e| \ll v_i < v_e$ is also satisfied. In general, as previously seen in Sec. II E, the reference electron velocity is $\max(v_e, v_F)$, and typically $v_i < v_e \ll v_F$ when electrons are degenerate. The formulas (14) and (17) are obtained assuming $|\mathbf{u}_i - \mathbf{u}_e| \ll v_i$, instead of $|\mathbf{u}_i - \mathbf{u}_e| \ll v_F$, as one could wrongly assume from a direct extension of usual results.
[40]L. Spitzer, *Physics of Fully Ionized Gases* (Interscience, New York, 1956).
[41]J. Daligault and G. Dimonte, Phys. Rev. E **79**, 056403 (2009), and references therein.
[42]A. M. Dimits and B. I. Cohen, Phys. Rev. E **49**, 709 (1994).
[43]Z. Lin, W. M. Tang, and W. W. Lee, Phys. Plasmas **2**, 2975 (1995).
[44]P. J. Catto and K. T. Tsang, Phys. Fluids **20**, 396 (1977).
[45]X. Q. Xu and M. N. Rosenbluth, Phys. Fluids B **3**, 627 (1991); see Appendix A.
[46]H. Grad, Commun. Pure Appl. Math. **2**, 325 (1949).
[47]Additional information on orthogonal polynomials with respect to $f_0(1-\theta f_0)$ and their relation to the usual Hermite tensor polynomials can be found in Ref. 49.
[48]J. Daligault, "Trivariate orthogonal polynomials for quantum kinetic theory," J. Stat. Phys. (submitted).
[49]This property can be readily shows by combining the self-adjointness property (28) and the collisional invariants property (29). Indeed, with $a = c_0 + \mathbf{c}_1 \cdot \mathbf{p} + c_2\mathbf{p}^2$ and $b = \delta f/f_0(1-\theta f_0)$, (29) writes as $\int d\mathbf{p} a\hat{C} = [a,b]_{\tilde{0}} = 0$ for all b, which is equivalent to $[b,a]_{\tilde{0}} = \int d\mathbf{p} b f_0(1-\theta f_0)\tilde{\mathcal{O}}[a f_0(1-\theta f_0)] = 0\ \ \forall b$, and $\tilde{\mathcal{O}}[a f_0(1-\theta f_0)] = 0$. Here, $[a,b]_{\tilde{0}} = [a,b]_1 + [a,b]_{\tilde{2}}$.

Reuse of AIP Publishing content is subject to the terms at: https://publishing.aip.org/authors/rights-and-permissions. Downloaded to IP: 192.12.184.6 On: Tue, 22 Mar 2016 15:05:36